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## FOUNDATIONS, THEORY OF SETS, LOGIC

**Shepherdson, J. C. On the interpretation of Aristotelian syllogistic.** J. Symb. Logic 21 (1956), 137–147.

This paper is concerned with the representation theory of formalised Aristotelian syllogistic. It is shown that any model of Łukasiewicz' system of axioms for Aristotelian syllogistic, which is formulated in terms of the relations $Aab$, $Iab$ ("all $a$ are $b$", "some $a$ are $b$") can be represented by a set of non-empty proper subsets of a particular set, with the natural interpretation for $A$ and $I$. This representation is to be understood in the sense of a quasi-isomorphism, reducing to an isomorphism if $Aab$ and $Aba$ implies $a=b$. Similar results are obtained for axiomatic systems based on inclusion and complementation. {It might be interesting to correlate the contents of the present paper with Stone's representation theory for Boolean algebras.} *A. Robinson* (Toronto, Ont.).

**Takeuti, Gaisi. On the fundamental conjecture of GLC. I.** J. Math. Soc. Japan 7 (1955), 249–275.

The author continues his investigations of $G^1LC$ introduced in a previous paper (LC) [Jap. J. Math. 23 (1953), 39–96; 24 (1954), 149–156; MR 17, 701]. He considers here a subsystem in which the rules $(R_A, R_E)$,

$$\frac{F(V),\ A\to B}{(\phi)F(\phi),\ A\to B},\quad \frac{A\to B,\ F(V)}{A\to B,\ (E\phi)F(\phi)},$$

are restricted to those formulae $F(a)$ with free variable $a$ which do not contain bound variables of type 1. A modification of Gentzen's consistency proof for arithmetic [Math. Ann. 112 (1936), 493–565] is used to show that any theorem of this subsystem can be proved without cut: this is the "fundamental conjecture". {It seems to the reviewer that the author assumes in his analysis that the term $V$ in $R_A$, $R_E$ does not contain such bound variables either.} — A consequence of the restriction is that though a truth definition for arithmetic can be given in the notation of $G^1LC$, the definition requires variables of type 1 and hence induction cannot be applied unrestrictedly in this subsystem to formulae containing this truth definition. The significance of the cut theorem is different for the predicate calculus of first order, for which it was established by Gentzen [Math. Z. 39 (1934), 176–210, 405–441], and for GLC: in the former the rules are such that a logical symbol, which has been introduced, cannot be eliminated, in the latter it can, e.g. the logical symbols of $V$ are eliminated by application of $(R_A, R_E)$. Thus the consistency of GLC does not follow quite trivially from the fundamental conjecture, but depends on the fact that whenever a logical symbol is eliminated, a new bound variable is introduced, and hence theorems without quantifiers are identically true. A more detailed study of the axiom of infinity [$\Delta$ in the review of (LC)] shows that the (hypothetical) cut theorem for GLC implies the consistency of GLC with $\Delta$ as its axiom.
*G. Kreisel* (Princeton, N.J.).

**Takeuti, Gaisi. On the fundamental conjecture of GLC. II.** J. Math. Soc. Japan 7 (1955), 394–408.

The author considers the subsystem of $G^1LC$ [see the preceding review] in which the rules $(R_A, R_E)$ are restricted to terms $V$ which are not of type 0. The proof of the cut theorem uses only ordinary induction. The author does not give any indication of the intuitive content of the system considered. *G. Kreisel* (Princeton, N.J.).

**Takeuti, Gaisi. On the fundamental conjecture of GLC. IV.** J. Math. Soc. Japan 8 (1956), 145–155.

Yet another subsystem of $GLC^1$ [Jap. J. Math. 23 (1953), 39–96; 24 (1954), 149–156; MR 17, 701] for which the cut theorem holds. The system is a slight extension of one considered in the paper reviewed second above.
*G. Kreisel* (Princeton, N.J.).

**Shimauti, Takakazu. Proof of a special case of the fundamental conjecture of Takeuti's GLC.** J. Math. Soc. Japan 8 (1956), 135–144.

A subsystem of $GLC^1$ [see the preceding review] is described for which the cut theorem is easily established; apparently this fact constitutes the only reason for studying this system. The author shares Takeuti's opinion [Jap. J. Math. 23 (1953), 39–96; 24 (1954), 149–156; MR 17, 701] that analysis would be consistent if every theorem of GLC were provable without a cut, but he does not give a detailed proof. *G. Kreisel* (Princeton, N.J.).

**Markwald, Werner. Ein Satz über die elementar-arithmetischen Definierbarkeitsklassen.** Arch. Math. Logik Grundlagenforsch. 2 (1956), 78–86.

An arithmetical predicate $P$ is partially axiomatizable if there is either a constructive process for determining infinitely many numbers for which $P$ holds, or one for determining infinitely many numbers for which $P$ fails. This paper considers predicates which are partially axiomatizable in the above sense, though not axiomatizable in the sense of Kleene [Introduction to metamathematics, Van Nostrand, New York, Amsterdam, 1952, § 62; MR 14, 525]. Let $\mathfrak{D}$ be an arbitrary sequence made up of the quantifier symbols $\forall$ and $\exists$. A set $M$ is in the class $\mathfrak{D}$ if it is defined by a predicate of the form $\exists x_1 \cdots \forall x_n R(x_0, x_1, \cdots, x_n)$, where the sequence of quantifiers in the prefix is the same as that in $\mathfrak{D}$. With respect to $\mathfrak{D}$, the operations $\bar{\mathfrak{D}}$ (the result of interchanging $\forall$'s and $\exists$'s in $\mathfrak{D}$), and $(\mathfrak{D})$ (the set field generated by the elements of $D$) are defined. A function $f$ is called an $(\forall)$-Function if it is defined by the scheme

$$f(a, 0)=b$$

$$f(a, n+1)=\begin{cases}\alpha(n, f(n)) \text{ if } P(a, n, f(n)),\\ \beta(n, f(n)) \text{ otherwise,}\end{cases}$$

where $P \in (\forall)$, and $\alpha$ and $\beta$ are partial recursive.

The main theorem states that there is a set $M$ in

$\exists\forall\cup\forall\exists$ such that neither $M$ nor $\bar{M}$ contains an infinite enumerable subset. It is shown that ($\forall$)-Predicates are partially axiomatizable. The theorem of Mostowski [Ann. Soc. Polon. Math. 21 (1948), 114–119; MR 10, 175] follows as a corollary. These concepts and results are extended to ($\forall\mathfrak{D}$)-Functions, and a generalization of the Mostowski theorem is formulated and proved.

*E. J. Cogan* (Hanover, N.H.).

**Kondô, Motokiti. Sur la notion du transfini.** C. R. Acad. Sci. Paris **242** (1956), 2209–2212.

This note is based on the author's previous work [same C.R. **242** (1956), 1841–1843, 1945–1948, 2084–2087; MR **17**, 933]. In order to introduce $S$-namable ordinals by means of concepts of elementary arithmetic, he considers $S$-namable sets of rationals which are well-ordered by their natural ordering. The work is said to be based on two lemmata: (i) If $K_0$ and $K$ are relative continua [cf. review of C.R. Acad. Sci. Paris **242** (1956), 1945–1948] two $(S, K_0, K)$-namable sets of rationals in natural order are $(S, K_0, K)$-isomorphic if they are isomorphic at all, (ii) an $(S, K_0, K)$-set $E$ of rationals is well-ordered in natural order if and only if each point of $K_0$ is isolated on the left in $E$. [The author's statement of (ii) contains an obvious misprint.] By means of these ordinals a transfinite hierarchy of namable sets is introduced. — On the author's own definition of a relative continuum (r.c.) on p. 1948 both lemmata are false. For, since every ordering of the integer is $(S, K_0, K_0)$-isomorphic to some $(S, K_0, K_0)$-set of rationals in natural order, for $K_0=\pi(R)$, a counter example to (i) is obtained from the two recursive orderings of order $\omega^\omega$ mentioned at the end of the review of Markwald [Math. Ann. **127** (1954), 135–149; MR **15**, 771], and a counter example to (ii) is obtained from the ordering $>^R$ on p. 422 of Kleene [Amer. J. Math. **77** (1955), 405–428; MR **17**, 5], which is not a well-ordering, but all descending $(S, K_0, K_0)$-sequences are finite. If lemma (ii) is correct on the definition of r.c. in the review of Kondô, C.R. Acad. Sci. Paris **242** (1956), 2084–2087 [MR **17**, 933] the reals definable in Kleene's hyperarithmetic hierarchy do not constitute a r.c. *G. Kreisel* (Princeton, N.J.).

**Kondô, Motokiti. Sur le continu projectif et la conclusion de l'étude des ensembles nommables.** C. R. Acad. Sci. Paris 242 (1956), 2275–2278.

In this note the projective closure of the set of rationals $R$ is considered. The author states that it is obtained by means of the transfinite hierarchy [see the preceding review] based on $(S, \pi(R), C)$-definable ordinals, where $C$ denotes the continuum. Some comments are made on the relation between the author's ideas and the model $\Delta$ of Gödel's tract [Consistency of the continuum hypothesis, Princeton; 1940; MR **2**, 66]. *G. Kreisel.*

**Skolem, Th. A version of the proof of equivalence between complete induction and the uniqueness of primitive recursion.** Norske Vid. Selsk. Forh., Trondheim **29** (1956), 10–15.

Let $(S)$ denote the usual system of primitive recursive arithmetic, namely (i) the notation and rules of the propositional calculus, (ii) the axiom schema for equality, (iii) the (axiom) schema for primitive recursive "defining" equations, (iv) for induction: if $A(0)$, and $A(n)\to A(n+1)$ have been proved, $A(n)$ is inferred. Let $(E)$ denote the "pure-equation arithmetic" in which no logical symbols occur, equality, its only predicate symbol, not being counted as logical, and the rules of proof are (ii') $n=n$, if $n=m$ has been proved, $m=n$ and $f(n)=f(m)$ are inferred, (iii) and (iv'): if, for some $H$, $f(0)=g(0)$, $f(n+1)=H[n, f(n)]$, $g(n+1)=H[n, g(n)]$ have been proved then $f(n)=g(n)$ is inferred. $(S)$ and $(E)$ are equivalent in the two senses: (a) Exactly the same equations are provable in (S) and $(E)$, (b) with any formula $A$ of $(S)$ is associated an equation $t_A=0$ where $t_A$ is a (primitive recursive) term of $(E)$, and hence of $(S)$, $\vdash_S(A\leftrightarrow t_A=0)$, $t_{A\vee B}$ is $t_A\cdot t_B$, $t_{A\&B}$ is $t_A+t_B$ and if $A$ is of the form $t=0$ then $t_A$ is $t$, so that $t_A=0$ "expresses" $A$ in $(S)$; then $\vdash_S A$ if and only if $\vdash_E t_A=0$. This result, which is due to Bernays, first appeared in Goodstein [Proc. London Math. Soc. (2) **48** (1945), 401–434; MR **8**, 245]. A closely related result was published by Bernays [Kontrolliertes Denken, Alber, München, 1951, pp. 10–17]. The author gives a simple exposition of the main steps in the equivalence proof without giving a precise formulation of the result. His argument is very similar to Bernays' 1951 paper, which, apparently, the author does not know.

{Reviewer's note. Th. 10 (p. 14) is tacitly assumed in the proof of Th. 8 (p. 13): this does not spoil the paper since Th. 10 is proved without the use of Th. 8. — Replace the last occurrence of "$f$" in line 9 of p. 11 by "$g$".}

*G. Kreisel* (Princeton, N.J.).

**Goodstein, R. L. A constructivist theory of plane curves.** Fund. Math. **43** (1956), 23–35.

"This paper develops a theory of $p$-curves, which are finite matrices with binary fraction elements. Roughly speaking, a $p$-curve is a finite assemblage of points in serial order on a grid, the jump from one point to the next being of fixed amount in one or other of two "directions". The concept of a plane curve is then introduced in terms of sequence of $p$-curves. The emphasis throughout the paper is on the strictly finitist character of the proof processes." (Author's introduction.)

*G. Kreisel* (Princeton, N.J.).

**Słomiński, J. On the extending of models. III. Extensions in equationally definable classes of algebras.** Fund. Math. 43 (1956), 69–76.

[For parts I–II see Łoś, Fund. Math. **42** (1955), 38–54; Łoś and Suszko, ibid. **42** (1955), 343–347; MR **17**, 224, 815.] Pták [Čehoslovack Mat. Ž. **2(77)** (1952), 247–271; MR **15**, 598] gave an "algebraic" version of Malcev's characterization [Mat. Sb. N.S. **6(48)** (1939), 331–336; **8(50)** (1940), 251–264; MR **2**, 7, 128], by means of universal sentences, of those semigroups which can be extended to a group. The author points out that Pták's argument applies to several equationally definable classes of algebras. But the only applications which he makes of this observation are the prehistoric theorems: each distributive structure can be extended to a Boolean algebra, and each semigroup can be extended to a ring.

*G. Kreisel* (Princeton, N.J.)

**Šanin, N. A. Some problems of mathematical analysis in the light of constructive logic.** Z. Math. Logik Grundlagen Math. **2** (1956), 27–36.

The author sketches how an analogue to the theory of Lebesgue measurable linear sets can be constructed on the basis of constructive analysis in the sense of Specker [J. Symb. Logic **14** (1949), 145–158; MR **11**, 151]. His method is roughly as follows. He uses "real number" in the sense of Specker's "berechenbare reelle Zahl"; also "sequence" and "convergent" must be understood in the constructive sense. Thus every real number is defined

by a general recursive function and can be characterized by its Gödel number. A sequence of real numbers is a general recursive function, which has Gödel numbers of real numbers as values. Convergence of such a sequence is defined analogously to Specker's Def. II. A finite set of rational open intervals and rational points is called a complex. Convergence of a sequence of complexes is defined on the basis of the measure of the symmetrical difference. A convergent sequence of complexes is called a measurable set (of finite measure). As every measurable set is given by a general recursive function, it can be characterized by the Gödel number of that function. The elementary operations of union, etc. of measurable sets are given by partial recursive functions, operating on the Gödel numbers. As for real numbers, the notions of a sequence of measurable sets and of a convergent sequence of measurable sets can be introduced. However, it follows easily from Specker's Satz I, that if an extending convergent sequence of measurable sets is given, there does not always exist a measurable set, which is the union of the members of the sequence. *A. Heyting.*

**Ackermann, Wilhelm. Zur Axiomatik der Mengenlehre.** Math. Ann. 131 (1956), 336–345.

In this paper is introduced a set theory formalized within the lower predicate calculus with the usual epsilon relation and a unary predicate $M$ accepted as primitive. The predicate $M$ is to be interpreted as asserting for argument $x$ that $x$ is a set. The axioms for the theory, in addition to the extensionality axiom, are:

$$(x)(\mathfrak{A}(x)\rightarrow M(x)\rightarrow(Ey)(z)(z\in y\leftrightarrow\mathfrak{A}(z)),$$

$$(Mx_1\&\cdots\&Mx_n)\rightarrow((y)(\mathfrak{A}_0(y)\rightarrow My)\rightarrow (Ez)(Mz\&(u)(u\in z\leftrightarrow\mathfrak{A}_0(u))),$$

$$(Mx\&(y\in x\vee y\subset x))\rightarrow My,$$

where $\mathfrak{A}(x)$ is any formula of the theory containing the variable $x$ free, $\mathfrak{A}_0(y)$ is any formula of the theory in which $M$ does not appear and in which $y$ is free, and $x_1, \cdots, x_n$ are all free variables (if any) of $\mathfrak{A}_0(y)$ different from $y$. That this theory is non-trivial is shown by deriving all the axioms of the set theory of Fraenkel, with the exception of the axiom of choice, as theorems of the theory. Russell's paradox, as a typical syntactical paradox, is briefly discussed. *P. C. Gilmore.*

**Sprinkle, H. D. A development of cardinals in "The consistency of the continuum hypothesis".** Proc. Amer. Math. Soc. 7 (1956), 289–291.

The object is to obtain a theory of cardinals based on the development in Gödel's book of the title [Princeton, 1940; MR 2, 66] but without the axiom of choice; this is done by defining cardinals for ordinals only, rather than for arbitrary sets. The cardinal $\bar{X}$ of an ordinal $X$ is defined as follows:

$$\alpha\simeq\bar{\alpha}\cdot\bar{\alpha}\in On\cdot(\beta)(\beta<\bar{\alpha}\cdot\supset\cdot\sim(\beta\simeq\alpha))\cdot\overline{On}=On;$$

thus the cardinal $\bar{\alpha}$ of an ordinal number $\alpha$ is the smallest ordinal equivalent to $\alpha$. Then (1) there exists a monotone function on $On$ that takes $\alpha$ to $\bar{\alpha}$; (2) the class of all infinite cardinals is a proper class, and is isomorphic to $On$ with respect to the $\varepsilon$-relation; (3) for every set $x$, the class of ordinals whose cardinal is a member of $x$ is a set. *L. Gillman* (Lafayette, Ind.).

# ALGEBRA

## *Combinatorial Analysis*

**Erdös, P.; and Rényi, A. On some combinatorical problems.** Publ. Math. Debrecen 4 (1956), 398–405.

Let $C_k(n)$ denote the least number of combinations of $k$ out of $n$ elements ($k, n=2, 3, \cdots$) such that every pair of elements occurs at least once in a combination. Clearly $C_k(n)\geq n(n-1)/[k(k-1)$ and in case of equality the combinations form a balanced incomplete block design with $\lambda=1$.

The authors investigate the asymptotic behaviour of $C_k(n)$. Putting $C_k(n)k(k-1)/[n(n-1)]=c_k(n)$ the authors prove 1) $\lim_{n\to\infty} c_k(n)$ exists for every $k$. 2) $\lim_{n\to\infty} c_k(n)=1$ if $k$ is a prime-power. 3) $\lim_{k\to\infty}\lim_{n\to\infty} c_k(n)=1$.

Let $D_k(n)$ denote the length of the shortest sequence formed from $1, 2, \cdots, n$ in which any two distinct digist are at least once in such a position that they are separated by at most $k$ numbers. Using the preceding results and a lemma of Szele [Mat. Fiz. Lapok **50** (1943), 223–256; MR 8, 284 the authors prove that $\lim_{n\to\infty} D_k(n)/n^2$ exists. *H. B. Mann* (Columbus, Ohio).

**Sade, Albert. Sur les substitutions dont les cycles sont ordonnés et sur les partitions.** Ann. Fac. Sci. Marseille 24 (1955), 67–81.

The author writes every permutation as a product of cycles without common elements in such a way that the smallest element in each cycle is written first and the cycles are arranged in order according to their first elements. Let $R(n, p)$ be the number of permutations of $n$ digits consisting of $p$ cycles, $g(n, k, c)$ and $F(n, k)$ resp. the number of such permutations where the $k$th cycle has length $c$ and where the $k$th cycle contains the digit $n$. The author computes these three functions by means of difference equations satisfied by them and by means of relations between them. He then studies relations between these functions and others derived from them and applies them to the theory of partitions. *H. B. Mann.*

**Carlitz, L.; and Riordan, J. The number of labeled two-terminal series-parallel networks.** Duke Math. J. 23 (1956), 435–445.

This paper deals with the number $S_{n,r}$, of two-terminal series-parallel networks with $n$ elements, $r$ of which are labeled with distinct labels. Introducing the generating function

$$S(y, z)=\sum_{r=0}^{\infty} y^r \sum_{r=0}^{\infty} S_{n+r,n}z^n/n!$$

the authors derive the fundamental relation

$$(1+S(y, z))\exp(S(0, z))=(1+S(0, z))(\exp\tfrac{1}{2}(S(y, z)+yz)).$$

From this other relations for $S$ and recurrences for $S_{n,r}$ are obtained. In particular, if $S(x, y)=\sum_{n=1}^{\infty} y^nS_n(x)$ and prime denotes differentiation then

$$S_n'(x)=S_{n-1}(x)+\sum_{k=1}^{n-1} S_{n-k}(x)S_k'(x).$$

This formula was derived earlier in unpublished work of R. M. Foster. The main concern in this paper is with arithmetic properties of $S_{n,r}$ and the principal result here is that for $p$ an odd prime

$$S_{n+m+w,m+w}\equiv 2^eS_{n+m,m} \pmod{p^{e+l}}$$

with $m \geqq e+1$, $w=p^{e+j}(p-1)$ and $p^{j-1} \leqq nMp^j$. The numbers $A_n=S_{n,n}$ receive special consideration.

*L. Moser* (Edmonton, Alta.).

**Hogben, Lancelot. Combinatory notation.** Nature **178** (1956), 329.

**Surányi, János. On a problem of old Chinese mathematics.** Publ. Math. Debrecen **4** (1956), 195–197.

The identity

$$\sum_{r=0}^{k} \binom{k}{r}^2 \binom{n+2k-r}{2k} = \binom{n+r}{r}^2$$

appeared without proof in a book of the Chinese mathematician Le-Jen Shoo of 1867; a proof was given by P. Turán [Mat. Lapok **5** (1954). 1–6; MR **16**, 13]. Additional proofs have been given by G. Szekeres, L. K. Hua, L. Takacs, G. Huszár and the reviewer. The present paper contains a combinatorial proof of the more general identity

$$(*) \qquad \sum_{r=0}^{k} \binom{k}{r}\binom{h}{r}\binom{n+k+h-r}{k+h} = \binom{n+k}{k}\binom{n+h}{h}.$$

{Incidentally (*) is a special case of Saalschütz's theorem.}

*L. Carlitz* (Durham, N.C.).

See also: Harary, p. 56; Mack, p. 63.

## *Linear Algebra, Polynomials, Invariants*

**Estermann, T. On the fundamental theorem of algebra.** J. London Math. Soc. **31** (1956), 238–240.

The author gives an elementary proof of a Lemma which occurs in some proofs of the Fundamental Theorem of Algebra; namely that, if $f$ is a non-constant polynomial and if $f(z_0) \neq 0$, then there exists a number $z$ such that $|f(z)| < |f(z_0)|$. His proof uses the function

$$g(w) = f(z_0+w)/f(z_0) = 1 + a_n w^n + \cdots + a_m w^m \ (a_n \neq 0, n \leqq m).$$

If $s$ is chosen so that $u = -\operatorname{Re}\{a_n s^n\} > 0$, then it can be shown that $|g(st)| \leqq 1 - (\frac{1}{2})ut^n$ and hence

$$|f(z_0+w)| < |f(z_0)|.$$

*M. Marden* (Milwaukee, Wis.).

**Thurston, H. S. On matric solutions of a cyclic equation.** Amer. Math. Monthly **63** (1956), 405–407.

It is proved that if $f(x)=0$ is a cyclic equation over the rational field there exists a conjugate set of rational matric solutions $B_i$ such that $B_{i+1}$ is obtained from $B_i$ by cyclicly permuting rows and columns.

*D. E. Rutherford* (St. Andrews).

**Palamà, G. Equazioni reciproche in senso generale.** Giorn. Mat. Battaglini (5) 3(83) (1955), 189–210 (1956).

An equation $f(x)=0$ of degree $n$ is said to be reciprocal (in the general sense) if $f(x)=Kx^n f(\lambda/x)$, where $K$ and $\lambda$ are suitable constants, $\lambda$ being called the modulus of reciprocity. The author confines himself to equations with real coefficients. The first step is to remove the roots $\sqrt{\lambda}$, $-\sqrt{\lambda}$ if they occur. The remaining equation is still reciprocal and is of even degree, $2k$, say. The substitution $y=x+(\lambda/x)$ then reduces the problem to solving an equation of degree $k$. Details are given of a class of equations of degree $m2^r$ which can be reduced to equations of degree $m$ by the successive substitutions $x_{i+1}=x_i+(\lambda_i/x_i)$ $(i=0, 1, 2, \cdots, r-1;\ x_0=x)$.

*W. Ledermann* (Manchester).

**Turán, Paul. Remark on the zeros of characteristic equations.** Publ. Math. Debrecen **4** (1956), 406–409.

Let the zeros of the characteristic equation

$$\begin{vmatrix} b_{11}-\lambda \cdots b_{1n} \\ \cdot \quad \cdots \quad \cdot \\ b_{n1} \quad \cdots b_{nn}-\lambda \end{vmatrix} = 0$$

lie in the strip $\beta_1 \leqq \Re z \leqq \beta_2$ and those of

$$\begin{vmatrix} a_{11}-\lambda \cdots & a_{1n} \\ \cdot \quad \cdots & \cdot \\ a_{n1} \quad \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

in $\alpha_1 \leqq \Re z \leqq \alpha_2$. Under the assumption of an upper estimation of $F = \sum_{j=1}^{n} \sum_{k=1}^{n} |b_{jk}-a_{jk}|^2$ the author gives a lower estimation of $\beta_1$ in terms of $n$, $F$ and the $a_{jk}$ numbers. The proof is performed via differential equations. The author remarks that a direct algebraical proof based on matrix calculus (which is easy when the $(a_{jk})$ and $(b_{jk})$ matrices are real and symmetrical) would be of interest.

*M. Zlámal* (Brno).

**Parodi, Maurice. Sur la localisation des valeurs caractéristiques des matrices dans le plan complexe.** C. R. Acad. Sci. Paris **242** (1956), 2617–2618.

Let $C=(c_i\delta_{ij})$ denote a diagonal matrix with all $c_i>0$. The author applies Gerschgorin's theorem [see, e.g., O. Taussky, Amer. Math. Monthly **56** (1949), 672–676; MR **11**, 307] to the matric $C^{-1}AC$. He thereby concludes that all eigenvalues $\lambda$ of $A$ lie in the union of the $n$ disks

$$(*) \qquad |c_{ii}-z| \leqq c_i^{-1} \sum_{j=1 (j \neq i)}^{n} |a_{ij}| c_j \ (i=1, \cdots, n).$$

Weaker bounds for the $\lambda$ arise from applying Hölder's inequality to (*). A numerical example of order 6 shows a case where (*) is more useful than Gerschgorin's theorem.

*G. E. Forsythe* (Los Angeles, Calif.).

**Jaeckel, K. Zum Eigenwertproblem normaler Matrizen.** Z. Angew. Math. Mech. **36** (1956), 150–151.

Let $A^*$ denote conjugate transpose (reviewer's notation). Assume $AA^*=A^*A$, i.e., that $A$ is normal. The author gives a short proof of the known theorem that $Ax=\lambda x$ implies $A^*x=\bar{\lambda}x$. From this follow immediately such standard theorems as that hermitian matrices have only real eigenvalues. Finally, the unitary equivalence of $A$ to a diagonal matrix can be proved readily.

*G. E. Forsythe* (Los Angeles, Calif.).

**Farahat, H. K.; and Mirsky, L. A condition for diagonability of matrices.** Amer. Math. Monthly **63** (1956), 410–412.

A matrix $A$ is diagonable if and only if, for every complex number $\omega$, the matrix $A-\omega I$ is what the authors call a group matrix; that is a matrix which belongs to some multiplicative group of matrices with suitable definition of unit and inverse elements.

*W. Ledermann* (Manchester).

**Benedicty, Mario. Una nuova forma normale per le matrici quasi abeliane.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **18** (1955), 602–608.

Le premier but de ce travail est celui de définir une forme canonique nouvelle pour les matrices quasi-Abéliennes; cette forme est la suivante:

$$\chi = \left\| \begin{matrix} \Delta^{-1} & 0 & x \\ 0 & u & \Phi_1 \\ 0 & 0 & \Psi_1 \end{matrix} \right\|.$$

Ici: 1) $\Delta$ est une matrice diagonale entière d'ordre $p$ dont les éléments principaux $d_1 d_2 \cdots d_p$ sont des nombres entiers positifs tels que $d_1=1$ et que $d_{h-1}$ divise $d_h$; 2) la matrice $X$ est de la forme

$$\left\| \begin{matrix} O^{(\rho)} & G^{(\rho, p-\rho)} \\ G_{-1} & F^{(p-\rho)} \end{matrix} \right\|,$$

où $F$, $G$ sont des matrices complexes et $F$ est symétrique; 3) $\Phi_1$ est de la forme $\Phi_1{}^{(\delta_1, p)} = \| O^{(\delta_1, p)} \Phi^{(\delta_1, p-p)} \|$, où $\Phi$ est une matrice complexe arbitraire; 4) $\Psi_1$ est de la forme

$$\Psi_1{}^{(\delta_2, p)} = \left\| \begin{matrix} u^{(\rho)} & \Psi^{(\rho, p-\rho)} \\ O^{(\delta_2-\rho, \rho)} & O^{(\delta_2-\rho, p-\rho)} \end{matrix} \right\|,$$

où $\Psi$ est une matrice complexe arbitraire; 5) il faut ajouter la condition qu'il existe une matrice complexe $\Omega_{11}$, nécessairement symétrique, telle que la partie imaginaire de la matrice symétrique

$$\left\| \begin{matrix} \Omega_{11} & G+\Omega_{11}\Psi \\ (G+\Omega_{11}\Psi)_{-1} & F+\Psi_{-1}\Omega_{11}\Psi \end{matrix} \right\|$$

soit positive. Les nombres entiers $p$, $\delta_1$, $\delta_2$, $\varrho$ sont les nombres caractéristiques de $\chi$. Les relations qui expriment la symétrie de $F$ sont les seules relations qui passent entre les éléments de $F$, $G$, $\Psi$. Après cela on peut étudier les classes de matrices quasi Abéliennes normales en considérant comme équivalentes deux de telles matrices $\omega$, $\omega'$ lorsqu'il existe une matrice complexe $\alpha$, non dégénérée, et une matrice entière unimodulaire $\Lambda$ telles que $\omega'=\alpha\omega\Lambda$; c'est ce que l'A. fait dans la deuxième partie du travail.

*E. G. Togliatti* (Gênes).

**Bassaly, W. A. Application of the analysis of direct products of group representations to the complete reduction of tensor space.** Proc. Math. Phys. Soc. Egypt **5** (1954), no. 2, 17–26 (1955).

In a previous paper [same Proc. **4** (1952), no. 4, 105–116; MR **15**, 162] the author obtained complete reductions of third and fourth rank tensors by splitting the non-primitive idempotents into independent idempotents by means of standard Young tableau units. Here the analysis of the direct products of irreducible representations of the symmetric group $G_m$ and the corresponding irreducible representations of the full linear group $L_n$ are applied to split the principal idempotents into other primitive independent idempotents. Only papers of Murnaghan about 1938 are referred to.

*J. A. Schouten* (Epe).

**Kuiper, N. H.; and Yano, K. Two algebraic theorems with applications.** Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 319–328.

Les Auteurs donnent des conditions nécessaires et suffisantes pour que les tenseurs

$$v_i,\ h_{ji},\ T_{ji}^{\cdot\cdot h},\ \text{et}\ R_{kji}^{\cdots h} = -R_{jki}^{\cdots h}$$

soient invariants relativement au groupes orthogonal propre $O(n)$ des transformations linéaires de déterminant $+1$ et qui conservent le tenseur défini positif $g_{ji}$.

Ils en déduisent quelques applications à la théorie des espaces de Riemann munis d'une connexion linéaire.

Les auteurs donnent aussi des conditions nécessaires et suffisantes pour que les tenseurs précités soient invariants par rapport au sous-groupe $O(n-1)$ de $O(n)$ qui laisse fixe un vecteur unité donné.

*G. Papy* (Bruxelles).

**Sokolov, N. P. On pencils of real cubic ternary forms.** Izv. Akad. Nauk SSSR. Ser. Mat. **19** (1955), 201–232. (Russian)

The author gives a short theory of elementary divisors of three-dimensional matrices and applies the results to the classification of pencils of cubic forms. Some of the cases which correspond to the Segre notation [111], [1 1 1], [(11)1], [21], [(21)] and [3] are discussed in detail.

*E. M. Bruins* (Amsterdam).

**Gáspár, Julius. Eine axiomatische Theorie der kubischen Determinanten.** Publ. Math. Debrecen **4** (1956), 126–130.

The author defines a cubic determinant which extends in a natural way the definition for the usual two-dimensional determinant — this was previously done by L. H. Rice [J. Math. Phys. **9** (1930), 47–71]. He shows that if $\varphi(A)$ denotes the determinant of the three-dimensional matrix $A$, then (a) $\varphi(A+B)$ is a sum of determinants of all matrices each of whose "planes" is a corresponding plane of $A$ or $B$; (b) multiplying a matrix $A$ by a two-dimensional matrix $M$ multiplies $\varphi(A)$ by the determinant of $M$; (c) if $A$ is the identity matrix, then $\varphi(A)=1$.

*B. W. Jones* (Boulder, Colo.).

**Butson, A. T.; and Stewart, B. M. Systems of linear congruences.** Canad. J. Math. **7** (1955), 358–368.

The author gives a brief survey of the theory of linear equation and congruence over a principal ideal ring and then shows how this can be extended to systems over a set $Q$ of integral elements of an associative (but not necessarily commutative) algebra. The systems considered are of the form $\sum_{i=1}^{n} \alpha_{ij}\chi_i\beta_{ij}=\gamma_j$ $(j=1, \cdots, p)$ [or equality replaced throughout by congruence modulo ideals $M_j$] where $\alpha_{ij}$, $\beta_{ij}$, $\gamma_j$ are given elements of $Q$ and $\chi_i$ are unknowns. The question of solvability, the number of solutions and methods for obtaining the solutions are discussed by matrix methods.

*L. Fuchs.*

**Barna, Béla. Über die Divergenzpunkte des Newtonschen Verfahrens zur Bestimmung von Wurzeln algebraischen Gleichungen. II.** Publ. Math. Debrecen **4** (1956), 384–397.

This is a continuation of the author's [same Publ. 3 (1953), 109–118; 2 (1951), 50–63; MR **15**, 831; **13**, 18] and A. Rényi's [Mat. Lapok **1** (1950), 278–293; MR **12**, 321] earlier papers concerning the convergence of Newton's method of approximation. The author considers polynomials of degree $m$ $(m>4)$ with real zeros $\xi_1, \xi_2, \cdots, \xi_m$. It is first shown that such a polynomial can have a divergence interval, i.e., an interval which contains only irregular (divergence) points of the iteration. Then the author constructs a set $Z$ which has the following property: Every neighborhood of a point of $Z$ (or of its derivative set $Z'$) contains convergence points $a_1, \cdots, a_m$ such that Newton's iterative process starting with $a_i$ converges to the root $\xi_i$ $(i=1, 2, \cdots, m)$. Thus, the second problem raised in Rényi's paper [l.c.] is solved and one of his results is generalized. Finally, an arbitrary point of the complex plane is chosen as the starting point of the iteration and it is shown that the set of singular points has no interior point.

{Supplementing his earlier reviews this reviewer wishes to remark that results closely related, and partly overlapping, with this series of papers were also obtained by S. Gorn [Ann. of Math. (2) **59** (1954), 463–476; MR **15**, 781}.

*E. Lukacs* (Washington, D.C.).

**See also:** Walsh, p. 32; Herzog and Piranian, p. 26; Speranza, p. 64.

### *Partial Order Structures*

**Dean, Richard A. Component subsets of the free lattice on $n$ generators.** Proc. Amer. Math. Soc. **7** (1956), 220–226.

Let FL($n$) be the free lattice on $n$ generators. The author proves the following theorems. (1) Given any two words, unequal in FL($n$), there exists a finite homomorphic image of FL($n$) in which their images are distinct. (2) Any lattice possessing a countable number of generators can be embedded in a lattice with three generators. The last result was published earlier by Sorkin [Dokl. Akad. Nauk SSSR (N.S.) **95** (1954) 931–934; MR **15**, 926]. In the proof of (2) the concepts of component and of component subset are used. The components of a word in FL($n$) are defined recursively: (i) The only component of a generator is itself, and (ii) if $W \equiv A \cup B$ (or $A \cap B$) the components of $W$ are $W$, $A$, and $B$ and their components. A component subset $P$ of FL($n$) is a collection of words in FL($n$) with the property that if a word belongs to $P$ then so do all its components. *M. Novotný* (Brno).

**Pierce, R. S. Radicals in function rings.** Duke Math. J. **23** (1956), 253–261.

Ein Verbandring $R$ ist ein Ring, der gleichzeitig ein Verband ist, derart dass die Abbildungen $x \to x+a$ Verbandsantomorphismen sind und die Abbildungen $x \to ax$ und $x \to xa$ Ordnungsendomorphismen für $a \geqq 0$. Gilt für $a \wedge b = 0$ und $x \geqq 0$ stets $xa \wedge b = ax \wedge b = 0$, so heisst $R$ ein Funktionenring.

Jeder Funktionenring ist als subdirekte Summe von totalgeordneten Ringen darstellbar. Dieser und viele verwandte seihe werden mithilfe eines sehr allgemeinen Radikalbegriffs bewiesen. Die Forderungen, die an das Radikal $\varrho(R)$ von $R$ gestellt werden sind:

$$\text{(I) } \varrho(R/\varrho(R)) = 0, \quad \text{(II) } \varrho(R) = \bigcap_h h^{-1}\varrho(hR),$$

wenn $h$ alle Hormorphismen von $R$ auf total-geordnete Ringe durchläuft. *P. Lorenzen* (Bonn).

**Klein-Barmen, Fritz. Multiplikative Funktionen über euklidischen und Sternverbänden.** J. Reine Angew. Math. **195** (1955), 121–126 (1956).

A "multi-function" on a lattice $L$ is defined essentially as one such that $\phi(0)=1$, and $\phi(x \cup y) = \phi(x)\phi(y)$. A Mobius inversion formula is constructed for the "summator" $S\phi(x) = \sum \phi\,(Vp_\nu{}^\lambda)$, in any "star lattice". Since, in a star lattice, any initial segment $[O, a]$ is isomorphic with the set of divisors of some positive integer $n$, the generalization is almost immediate. *G. Birkhoff.*

**Hartmanis, Juris. Two embedding theorems for finite lattices.** Proc. Amer. Math. Soc. **7** (1956), 571–577.

The author defines a geometry $G$ on a set $S$ to be a collection of subsets such that any two distinct elements are contained in one and only one subset and every subset contains at least two distinct elements. For two geometries on $S$ one writes $G \supset H$ if for every line $l$ in $H$ there is a line $L$ in $G$ such that $L \supset l$. Then the collection of all geometries on $S$ form a complete point lattice. The main result is that any finite lattice can be imbedded in such a lattice of geometries for a finite set $S$. *O. Ore.*

**Petresco, J. Théorie relative des chaînes.** Publ. Sci. Univ. Alger. Sér. A. **2** (1955), 61–135 (1956).

In the first section of this paper the author examines the chains in lattices with the object of determining the most general conditions under which the analogies of the theorems of Jordan, Schreier, and Zassenhaus are valid. In the second section the various types of normality yielding these results are constructed and used to classify the normality concepts introduced by Uzkow [Mat. Sb. N.S. **4(46)** (1938), 31–43], Kořínek [cf. MR **7**, 509; **13**, 1138; **12**, 667; **13**, 525, 7], Livšic [cf. MR **10**, 674; **12**, 583] and the reviewer [Trans. Amer. Math. Soc. **41** (1937), 266–275]. *O. Ore* (New Haven, Conn.).

### *Rings, Fields, Algebras*

**Narita, Masao. On the structure of complete local rings.** J. Math. Soc. Japan **7** (1955), 435–443.

Généralisation et simplification des théorèmes de I. S. Cohen sur la structure des anneaux locaux complets. Soit $\mathfrak{o}$ un anneau commutatif admettant un unique idéal maximal $\mathfrak{m}$; supposons $\mathfrak{o}$ séparé et complet pour la topologie définie par les $\mathfrak{m}^n$; posons $F = \mathfrak{o}/\mathfrak{m}$ et notons $h$ l'homomorphisme canonique de $\mathfrak{o}$ sur $\mathfrak{o}/\mathfrak{m}$. On suppose que $F$ est de caractéristique $p \neq 0$, et on note $(x_\sigma)$ une famille d'éléments de $\mathfrak{o}$ tels que les $h(x_\sigma)$ forment une $p$-base de $F$ sur $F^p$. Supposons d'abord que l'on a $\mathfrak{m}^n = (0)$ pour certain entier $n$; pour tout élément $z$ de $F^{p^n}$, on écrit $z = (h(a))^{p^n}$ $(a \in \mathfrak{o})$ et alors $a^{p^n}$ est un élément $g(z)$ ne dépendant que de $z$; quand $\mathfrak{o}$ est de caractéristique $p$ (resp. 0 ou $p^q$ avec $q > 1$), $g(F^{p^n})$ (resp. $g(F^{p^{2n}}) + pg(F^{p^{2n-1}}) + \cdots + p^{n-1}g(F^{p^{n+1}})$) est un sous anneau $A$ de $\mathfrak{o}$, et $R = A[(x_\sigma)]$ est un sous corps de $\mathfrak{o}$ (resp. un sous anneau complet de $\mathfrak{o}$ d'idéal maximal $(p)$) tel que $h(R) = F$. Dans le cas général où aucun $\mathfrak{m}^n$ n'est nul, on obtient, par le procédé précédent, un sous anneau $R_n$ de $\mathfrak{o}/\mathfrak{m}^n$ pour chaque $n$, et, si l'on note $h_{n,n+1}$ l'homomorphisme canonique de $\mathfrak{o}/\mathfrak{m}^{n+1}$ sur $\mathfrak{o}/\mathfrak{m}^n$, on a $h_{n,n+1}(R_{n+1}) = R_n$; alors la limite projective $R$ des $R_n$ est un sous corps de $\mathfrak{o}$ (resp. un sous anneau complet de $\mathfrak{o}$ d'idéal maximal $(p)$) tel que $h(R) = F$. Compléments sur l'unicité de $R$ (quand $F$ est parfait) et sur sa détermination à un isomorphisme près (quand $F$ est imparfait). *P. Samuel.*

**Mori, Yoshiro. On the integral closure of an integral domain. II.** Bull. Kyoto Gakugei Univ. Ser. B. **7** (1955), 19–30.

Si $A$ est un anneau d'intégrité local complet, sa clôture intégrale $A'$ est un $A$-module de type fini (démonstration utilisant les théorèmes de structure de I. S. Cohen); on en déduit une démonstration simple du fait que, si $A$ est un anneau d'intégrité noethérien de dimension finie, sa clôture intégrale $A'$ est un anneau normal (=„endlich diskret Hauptordnung"); ceci comble les lacunes d'une démonstration antérieure de l'auteur [Mem. Coll. Sci. Univ. Kyoto, Ser. A. Math. **27** (1953), 249–256; MR **15**, 392]. L'utilisation des théorèmes de Cohen permet aussi de montrer que, si $A$ est un anneau d'intégrité local de dimension 2, sa clôture intégrale $A'$ est un anneau noethérien: on se réduit au cas local, on montre que tout idéal premier de $A'$ a un système fini de générateurs, et on utilise le fait (démontré par Cohen) que ceci implique que $A'$ est noethérien. La conclusion n'est plus valable si $\dim(A) \geqq 3$ [cf. Nagata, ibid. **28** (1954), 121–124; MR **16**, 107]; l'auteur donne diverses conditions nécéssaires et suffisantes pour que $A'$ soit alors noethérien. Il donne aussi diverses conditions nécéssaires et suffisantes pour que le complété $\hat{A}$ d'un anneau d'intégrité local $A$ n'ait pas d'élément nilpotent; la plus frappante est qu'il

existe $a$ dans $A$ tel que $\bar{A}a$ soit intersection d'idéaux premiers. *P. Samuel* (Clermont-Ferrand).

**Kleinfeld, Erwin. Generalization of a theorem on simple alternative rings.** Portugal. Math. 14 (1956), 91–94.

The author has previously proved [Amer. J. Math. 77 (1955), 725–730; MR 17, 231] that if an alternative ring has no proper two-sided ideals and is not nil then it is either associative or a Cayley-Dickson algebra over its center. The present paper generalizes this result by showing that if the intersection of the non-zero ideals of an alternative ring is not nil, (and hence not zero), then the ring is again either associative or a Cayley-Dickson algebra. An example, due to Kaplansky, is cited by the author to show that it is not enough to assume that the intersection of the non-zero ideals is not zero.

*R. L. San Soucie* (Eugene, Ore.).

**Nobusawa, Nobuo. On compact Galois groups of division rings.** Osaka Math. J. 8 (1956), 43–50.

Soient $D$ un corps (commutatif ou non), $G$ un groupe d'automorphismes de $D$, $\phi$ le corps des invariants de $G$; on considère surtout des groupes $G$ réguliers, c'est-à-dire tels que pour tout élément $s\neq 0$ du commutant de $\phi$ dans $D$, l'automorphisme intérieur $x\to sxs^{-1}$ appartienne à $G$. On dit que $G$ est localement fini si pour tout $x\in D$, l'ensemble des transformés de $x$ par $G$ est fini. L'auteur commence par donner une autre démonstration d'un résultat de Nagahara et Tominaga [Proc. Jap. Acad. 31 (1955), 655–658; MR 17, 578] d'après lequel un groupe régulier et localement fini $G$ ne peut contenir qu'un nombre fini d'automorphismes intérieurs (et plus précisément le commutant de $\phi$ est un corps fini s'il y a au moins un automorphisme intérieur distinct de l'identité). Inversement, si $G$ est régulier, ne contient qu'un nombre fini d'automorphismes intérieurs, et si tout sous-corps de $D$ engendré par $\phi$ et un autre élément de $D$ est de rang fini sur $\phi$, alors $G$ est localement fini. Le reste de l'article repose sur un „lemme 4" qui est trivialement faux tel quel (prendre pour $G$ le groupe réduit à l'identité); l'auteur a sans doute omis de supposer que $G$ est le groupe de tous les $\phi$-automorphismes de $D$, mais même avec cette hypothèse sa démonstration n'a pas de sens et il semble que pour lui en donner un il faudrait savoir que si $\Sigma\supset\phi$ est un sous-corps de $D$ de rang fini sur $\phi$, tout $\phi$-isomorphisme de $\Sigma$ dans $D$ se prolonge en un automorphisme de $D$ (ce que l'auteur „démontre" ensuite, en s'appuyant sur le „lemme 4"!). Les th. 4 et 5 sont trivialement faux, comme le montre la théorie de Galois classique, et le th. 6 s'appuie lui aussi sur le „lemme 4". *J. Dieudonné.*

**Utumi, Yuzo. On quotient rings.** Osaka Math. J. 8 (1956), 1–18.

If $R$ is a subring of $S$, then $S$ is called a (left) quotient ring of $R$ ($R\leqq S$) if for every pair of nonzero elements $x, y\in S$ there exists some $a\in R$ such that $ax, ay\in R$ and $ax\neq 0$. Using this more restrictive definition of a quotient ring than the reviewer's [Proc. Amer. Math. Soc. 2 (1951), 891–895; MR 13, 618], the author is able to construct the extended centralizer $\bar{R}$ of any ring $R$ having no total (right) divisors of zero and to show that $\bar{R}$ is the unique maximal quotient ring of $R$. It is also proved that $\bar{R}_n$ is the maximal quotient ring of the total matrix ring $R_n$. The author's definition and the reviewer's definition of a quotient ring coincide if the ring has a zero singular ideal. A left ideal $L$ of $R$ is called an $M$-ideal if $L/x\leqq R$ for $x\in R$ implies $x\in L$, where $L/x=\{y\in R;\ yx\in L\}$. It is shown that if $R\leqq S$, the sets of $M$-ideals of $R$ and $S$ are in a 1-1 correspondence with each other. If $R$ is a semisimple weakly reducible ring in the sense of Levitzki [Trans. Amer. Math. Soc. 74 (1953), 384–409; MR 14, 720], then $\bar{R}$ is both the left and the right maximal quotient ring of $R$, $\bar{R}$ has the same index as $R$, $\bar{R}$ satisfies the same polynomial identities as $R$, and $\bar{R}$ also is semisimple weakly reducible. *R. E. Johnson* (Northampton, Mass.).

**Levitzki, J. The matricial rank and its application in the theory of $I$-rings.** Univ. Lisboa. Revista Fac. Ci. A. (2) 3 (1954–1955), 203–237.

The author defines the matricial rank of a ring $S$, denoted $|S|$, by: $S$ contains a complete set of matrix units of degree $n$ but not of any higher degree. For an element $a\in S$, he defines the matricial rank $|a|$ by $|a|=|aS|$ $(=|Sa|)$ (the equality of $|aS|$ and $|Sa|$ is stated as a theorem). The following is then shown: $|ab|\leqq\min(|a|, |b|)$. He then proves that if $S$ is an $I$-ring ($I$-ring=Zorn ring= a ring in which every non-nil right ideal has an idempotent) and if $R_1$ and $R_2$ are right ideals of $S$, then $|R_1+R_2|\leqq|R_1|+|R_2|$. It then follows in case $S$ is an $I$-ring that $|a+b|\leqq|a|+|b|$ for any elements $a$, $b$ in $S$. Thus the elements in an $I$-ring which have finite matricial rank form a two sided ideal. He defines $E$-rings (which we do not define here), and uses the matricial rank results to obtain theorems about $E$-rings. He introduces $I_N$ rings (non-nilpotent right ideals have idempotents) and studies them. A homomorphism $\varphi: S\to S'$ is said to be stable if any complete set of matrix units $e_{ij}'$ in $S'$ has a complete set of matrix units $e_{ij}$ in $S$ as inverse images; an ideal is stable if it is the kernel of a stable homomorphism. (Every nil ideal is stable, for instance). He shows that the union of an ascending chain of stable ideals is a stable ideal and finds classes of stable ideals in rings.

*I. N. Herstein* (Philadelphia, Pa.).

**Jones, A.; and Lumer G. A note on radical rings.** Bol. Fac. Ingen. Agrimens. Montevideo 5 (1956), I–IV = Fac. Ingen. Agrimens. Montevideo. Publ. Didact. Inst. Mat. Estadist. 3 (1956), 11–15. (Spanish)

The main results of this paper are the following. (1) Every subring of a ring $R$ is a radical ring [in the sense of Perlis and Jacobson; see Jacobson, Amer. J. Math. 67 (1945), 300–320; MR 7, 2] if and only if for every $a\in R$, there is a polynomial $p(x)$ with integral coefficients and constant term 0 such that $p(a)=0$ and $p(-1)=-1$. (2) Every subalgebra of an algebra $A$ is a radical algebra if and only if every element of $A$ is nilpotent. (3) Every closed subalgebra of a radical Banach algebra is a radical algebra. *M. Henriksen* (Princeton, N.J.).

**Reiner, Irving. Maschke modules over Dedekind rings.** Canad. J. Math. 8 (1956), 329–334.

Let $\mathfrak{o}$ be a Dedekind ring, and let $\mathfrak{O}$ be an $\mathfrak{o}$-order in a separable, finite dimensional associative algebra over the quotient field of $\mathfrak{o}$. An $\mathfrak{O}$-module $M$ [assumed to be finitely generated and torsion free as an $\mathfrak{o}$-module] is called an $M_u$-$\mathfrak{O}$-module [an $(\mathfrak{O}, \mathfrak{o})$-injective $\mathfrak{O}$-module in the language of homological algebra] if it is an $\mathfrak{O}$-direct summand of every $\mathfrak{O}$-module which contains it as an $\mathfrak{O}$-submodule that is an $\mathfrak{o}$-direct summand. $M_o$-$\mathfrak{O}$-modules [$(\mathfrak{O}, \mathfrak{o})$-projective modules] are defined dually. The author proves that an $\mathfrak{O}$-module is an $M_u$-$\mathfrak{O}$-module if, for every prime ideal $\mathfrak{p}$ in $\mathfrak{o}$, the $\mathfrak{O}$-module it induces is an $M_u$-$\mathfrak{O}$-module, where $\mathfrak{O}$ denotes the $\mathfrak{o}/\mathfrak{p}$-algebra $\mathfrak{O}/\mathfrak{p}\mathfrak{O}$; the converse holds if $\mathfrak{O}$ is Frobenius over $\mathfrak{o}$, i.e.,

has dual $\mathfrak{o}$-bases. The analogous results for $M_{\mathfrak{o}}$-$\mathfrak{O}$-modules hold. *D. G. Higman* (Missoula, Mont.).

**Yaqub, Adil. On the theory of ring-logics.** Canad. J. Math. **8** (1956), 323–328.

L'auteur remarque que dans l'anneau $R=Z/(n)$ des entiers modulo $n$, il est possible d'exprimer $x+y$ par une combinaison des opérations $xy$, $1+x$ et $x-1$.

*J. Dieudonné* (Evanston, Ill.).

**Rees, D. Valuations associated with ideals. II.** J. London Math. Soc. **31** (1956), 221–228.

[Pour partie I voir Proc. London Math. Soc. (3) **6** (1956), 161–174; MR **17**, 1047.] L'auteur a démontré précédemment [ibid. **5** (1955), 107–128; MR **16**, 669] des cas particuliers du théorème suivant: soient $A$ un anneau noethérien, $\mathfrak{a}$ un idéal de $A$, $v_{\mathfrak{a}}(x)$ la fonction définie par $v_{\mathfrak{a}}(x)=n$ si $x \in \mathfrak{a}^n$ et $x \notin \mathfrak{a}^{n+1}$ et par $v_{\mathfrak{a}}(x)=\infty$ si $x \in \bigcap_{n=1}^{\infty} \mathfrak{a}_n$; on pose $v_{\mathfrak{a}}'(x)=\lim_n n^{-1}v_{\mathfrak{a}}(x^n)$; il existe alors un nombre fini de valuations $v_i$ de $A$, à valeurs entières ou $\infty$, et des entiers $e_i$ tels que $v_{\mathfrak{a}}'(x)=\min_i e_i^{-1}v_i(x)$, et que l'ensemble des $x$ tels que $v_i(x)=\infty$ soit un idéal premier minimal $\mathfrak{p}_i$ de $A$. Ce théorème est démontré ici sans aucune restriction. L'auteur simplifie d'abord la réduction au cas où $\mathfrak{a}$ est un idéal principal, puis traite le cas où $A$ est un anneau d'intégrité local, et passe enfin au cas général.

*P. Samuel* (Clermont-Ferrand).

**Rees, D. Valuations associated with a local ring. II.** J. London Math. Soc. **31** (1956), 228–235.

[Pour partie I voir Proc. London Math. Soc. (3) **5** (1955), 107–128; MR **16**, 669.] Avec les notations précédentes on démontre que, si $A$ est un anneau local dont le complété $\hat{A}$ n'a pas d'élément nilpotent, il existe un entier $t(\mathfrak{a})$ ne dépendant que le l'idéal $\mathfrak{a}$ tel que $0\leqq v_{\mathfrak{a}}'(x)-v_{\mathfrak{a}}(x)\leqq t(\mathfrak{a})$ pour tout $x\neq 0$ dans $A$. Comme dans sa démonstration du „lemme d'Artin-Rees" [Proc. Cambridge Philos. Soc. **52** (1956), 155–157; MR **17**, 576], l'auteur utilise l'anneau gradué $R(A, \mathfrak{a})$ des sommes formelles finies $\sum_{r=-p}^{q} c_r t^r$ où $c_r \in \mathfrak{a}^r$ ($t$: indéterminée), son anneau total de fractions $L(A, \mathfrak{a})$ (qui est un anneau local) et le complété $\bar{L}(A; \mathfrak{a})$ de ce dernier (qui est l'ensemble des sommes formelles infinies $\sum_{-\infty}^{+\infty} d_r t^r$ où $d_r \in \mathfrak{a}^r\hat{A}$). *P. Samuel.*

**Guérindon, Jean. Sur une famille d'équivalences en théorie des idéaux.** C. R. Acad. Sci. Paris **242** (1956), 2693–2695.

A correspondence between certain equivalence relations of ideals in a commutative noetherian ring $A$ and topologies of $A$ defined by means of the ideals of $A$ is set up. This is then used to derive various characterizations of unique factorization domains, integrally closed rings, and Dedekind rings. *A. Rosenberg* (Princeton, N.J.).

**Nagata, Masayoshi. On the chain problem of prime ideals.** Nagoya Math. J. **10** (1956), 51–64.

Etant donné un anneau $A$ notons $\dim(A)$ la plus grande des longueurs des chaines d'idéaux premiers de $A$. Considérons les conditions suivantes: ($C_1$) toute chaine maximale d'idéaux premiers de $A$ est de longueur $\dim(A)$; ($C_2$) pour tout idéal premier minimal $\mathfrak{p}$ de $A$, on a $\dim(A/\mathfrak{p})=\dim(A)$ et ($C_1$) est vraie pour tout anneau entier sur $A/\mathfrak{p}$; (C′) la condition ($C_1$) est vraie dans tout anneau $B$ qui est un $A$-module de type fini; (C″) la condition ($C_1$) est vraie dans tout anneau entier sur $A$. On voit que ($C_2$) équivaut à (C″) pour un anneau d'intégrité $A$, et à (C′) pour un anneau d'intégrité noethérien; pour un anneau semi-local complet $A$, ($C_1$) et ($C_2$) sont toutes deux équivalentes au fait que $\dim(A/\mathfrak{p})=\dim(A)$ pour tout idéal premier minimal $\mathfrak{p}$ de $A$. La question de savoir si (C′) et (C″) sont vraies pour un anneau d'intégrité local quelconque $A$ est restée longtemps ouverte; elle est résolue ici par la négative: au moyen de séries formelles algébriquement independantes l'auteur construit une suite ($A_j$) d'anneaux d'intégrité locaux noethériens tels que la clôture intégrale $A_j'$ de $A_j$ ait deux idéaux maximaux de hauteurs différentes; ainsi aucun $A_j$ ne vérifie ($C_2$); d'autre part $A_0$ vérifie ($C_1$), mais aucun autre $A_j$ ne vérifie cette condition. Ces anneaux $A_j$ sont des anneaux locaux non réguliers de multiplicité 1. L'auteur étudie aussi les anneaux "quasi-unmixed" (anneaux semi locaux noethériens dont les complétés vérifient ($C_1$)) et "unmixed" (anneaux locaux noethériens tels que, pour tout idéal premier $\mathfrak{p}$ de (0) dans $\hat{A}$, on ait $\dim(\hat{A}/\mathfrak{p})=\dim(A)$), ainsi que les propriétés de "permanence" de ces anneaux. Un appendice donne une élégante démonstration du résultat suivant, dû à I. S. Cohen: si tout idéal premier d'un anneau $A$ a un système fini de générateurs, alors $A$ est noethérien; un lemme montre que, si un idéal $\mathfrak{a}$ et un élément $x$ d'un anneau $A$ sont tels que $\mathfrak{a}+Ax$ et $\mathfrak{a}:Ax$ ont des systèmes finis de générateurs, alors il en est de même de $A$; on raisonne ensuite par l'absurde en considérant (grâce au th. de Zorn) un élément maximal de l'ensemble des idéaux de $A$ qui n'admettent pas de système fini de générateurs; un tel idéal n'est pas premier, et on lui applique le lemme. *P. Samuel* (Clermont-Ferrand).

**Moriya, Mikao. Zusammenhang zwischen Derivationsmodul und 2-Kohomologiegruppe. I.** J. Math. Soc. Japan **7** (1955), 444–452.

Let $k$ be a field that is complete for a discrete rank 1 valuation, and let $K$ be a separable finite algebraic extension of $k$. Let $r$, $R$ denote the valuation rings of $k$, $K$, resp., and let $P$ denote the valuation ideal in $R$. Let $H^n(R/r, R/P^m)$ stand for the $n$-dimensional cohomology group of the complex of the normalized $r$-linear symmetric $n$-cochains for $R$ in $R/P^m$. Using his earlier results on the structure of the $R$-modules $H^1(R/r, R/P^m)$ and $H^2(R/r, R/P^m)$ [Math. J. Okayama Univ. **2** (1953), 111–148; **5** (1955), 43–77; Proc. Japan Acad. **30** (1954), 787–790; MR **14**, 952; **17**, 578; **16**, 1087], the author shows here that, for $m$ large enough, these modules are $R$-isomorphic. He asserts that, actually, this result holds for all $m$, as he will show in a forthcoming paper.

*G. Hochschild* (Princeton, N.J.).

**Amitsur, S. A. Some results on central simple algebras.** Ann. of Math. (2) **63** (1956), 285–293.

This paper is the continuation of the author's previous article [Ann. of Math. (2) **62** (1955), 8–43; MR **17**, 9]. Here, using the same notions and methods as before, the author gives some complements to his theory, which lead to some applications in different fields of algebra. The contributions to the theory itself will not be mentioned here because their formulation involves too many notions that need a definition. In the first application, the author shows that the degrees of the representations of $\mathrm{GL}(n)$ obtained by decomposition of the tensor space of degree $m$ are all divisible by $n/(n, m)$. The second application indicates that the mapping $a\to a^{p^v}$ of a field of characteristic $p$ can be extended to a given central algebra over that field if and only if $v$ equals 1, modulo the exponent of the given algebra. The third application considers all the central simple algebras over a field $C$ which are split

together by a normal extension $F$ of $C$ and by another field $K$ (which contains $C$). The author proves that the Brauer group $B(F, K)$ of all those algebras is isomorphic to the first cohomology group

$$H^1((F\otimes K)^*/F^*, G),$$

where the stars indicate the multiplicative groups of regular elements and $G$ denotes the Galois group of $F$.

*G. Papy* (Brussels).

**Beyer, Gudrun. Erweiterungsproblem galoisscher Körper und Zerfall einfacher Algebren.** J. Reine Angew. Math. **195** (1955), 215–220 (1956).

Suppose that $\Omega/\Omega_0$ is a normal field extension with the Galois group $\mathfrak{g}$ which is the factor group $\mathfrak{G}/\mathfrak{N}$ so that the characteristic of $\Omega_0$ is relatively prime to the order of $\mathfrak{N}$. The author examines necessary conditions for the imbedding of $\Omega$ into a commutative Galois algebra $K/\Omega_0$ with the Galois group $\mathfrak{G}$ (i.e. the semi simple algebra $K/\Omega_0$ has a normal basis over $\Omega_0$, the elements of $\Omega_0$ are invariant with respect to $\mathfrak{G}$). For this purpose the crossed group ring $G$ of $\mathfrak{G}$ over $\Omega$ (with the elements $\sum Sa_S$, $S \in \mathfrak{G}$, $a_S \in \Omega$ and the defining relations $aS=Sa^S$, assuming the existence of $K/\Omega_0$) is considered as an extension of the group ring $N=\{\sum_{V\in N} Ua_U\}$. Suppose then that $N$ has the direct simple constituents $e_\chi/N$ belonging to the irreducible characters $\chi$ of $\mathfrak{N}$ in $\Omega$. Observing that the relations $(\sum_{U\in\mathfrak{N}} Ua_U)^S=S^{-1}(\sum Ua_U)S=\sum U^Sa_U^s$, $U^S=S^{-1}US$ and $s=S \bmod U$ lead to a permutation group $\Gamma$ of the idempotents $e_\chi\to e_\chi^s$ by $e_\chi^s=e_\chi$, which is homomorphic to $\mathfrak{g}$, and gives rise to domains of transitivity $\mathfrak{R}$ with respect to $\Gamma$. Let $e_\mathfrak{R}=\sum_{\chi\in\mathfrak{R}} e_\chi$. Then set $G=\sum_\mathfrak{R} e_\mathfrak{R}G$ with the simple components $e_\mathfrak{R}G$. It is shown that each direct component $e_\mathfrak{R}G$ is a crossed product of the Galois algebra $e_\mathfrak{R}N$ with the group $g$. If in particular $G=\bigcup_s R_sN$ with $R_sR_t=R_{st}Q_{s,t}$ ($Q_{s,t}\in N$) denotes a coset decomposition of $G$ over $N$ then the projections $r_s=e_\mathfrak{R}R_s$ with $\alpha r_s=r_s\alpha^{R_s}$, $\alpha\in e_\mathfrak{R}N$, $r_sr_t=r_{st}q_{s,t}$, $q_{s,t}=e_\mathfrak{R}Q_{s,t}$ form a basis of $e_\mathfrak{R}G$ over $e_\mathfrak{R}N$. Using results of Delone and Fadden [Mat. Sb. N.S. **15(57)** (1944), 243–284; MR **6**, 200], Hasse [J. Reine Angew. Math. **187** (1949), 14–43; MR **11**, 576], and Wolf [ibid. **193** (1954), 166–182; MR **16**, 790] necessary conditions for the imbedding of $\Omega/\Omega_0$ in an algebra $K/\Omega_0$ are found to be equivalent to the splitting of the factor set $q_{s,t}$, since the latter is equivalent to the existence of regular elements $B_s\in N$ with $B_{st}Q_{s,t}=B_tB_s^{R_t}$. Furthermore an analog to Hilbert's Theorem 90 is proved, showing that another solution $B_s'$ of the above equations satisfies $AB_s'=B_sA^{R_s}$ with regular $A\in N$.

*O. F. G. Schilling* (Chicago, Ill.).

**Reisel, Robert B. A generalization of the Wedderburn-Malcev theorem to infinite dimensional algebras.** Proc. Amer. Math. Soc. **7** (1956), 493–499.

Let $A$ be an algebra over a field $F$, with (Jacobson) radical $N$. If $\bigcap N^k=0$, then a Hausdorff topology, called the $N$-adic topology, can be imposed on $A$ by taking the sets $N^k$ ($k=1, 2, \cdots$) as a fundamental system of neighborhoods of zero. An algebra is called locally separable if every finite subset is contained in a finite dimensional separable subalgebra. Let $A$ be complete in the $N$-adic topology, and let $A/N$ be a locally separable algebra of at most countable dimension over $F$. Then $A$ contains a subalgebra $B$ such that $A=B+N$, and $B\cap N=0$. This result improves a theorem of the reviewer [Duke Math. J. **21** (1954), 79–85; MR **15**, 774] by removing a restriction on the dimension of $A/N$. The author's second result is a generalization of the Malcev theorem. Let $A$ be complete in the $N$-adic topology, and let $A/N$ be locally separable. For each positive integer $n$, let $N_n=N^n/N^{n+1}$ be complete with respect to the topology in which a fundamental system of neighborhoods of zero is furnished by the subsets of $N_n$ which are centralizers of finite dimensional separable subalgebras of $A/N$. If $B_1$ and $B_2$ are subalgebras such that $A=B_1+N=B_2+N$ and $B_1\cap N=B_2\cap N=0$, then $B_1$ is mapped onto $B_2$ by a quasi-inner automorphism of $A$ determined by an element of $N$. {An example shows that the second theorem is false unless some assumption besides completeness in the $N$-adic topology is made concerning the radical.}

*C. W. Curtis* (Madison, Wis.).

**Gillman, Leonard; and Henriksen, Melvin. Some remarks about elementary divisor rings.** Trans. Amer. Math. Soc. **82** (1956), 362–365.

**Gillman, Leonard; and Henriksen, Melvin. Rings of continuous functions in which every finitely generated ideal is principal.** Trans. Amer. Math. Soc. **82** (1956), 366–391.

The first of these papers is purely algebraic, and studies certain classes of commutative rings previously investigated by Helmer [Bull. Amer. Math. Soc. **49** (1943), 225–236; MR **4**, 185] and Kaplansky [Trans. Amer. Math. Soc. **66** (1949), 464–491; MR **11**, 155]. The classes include $F$-rings (i.e. all finitely generated ideals are principal), $T$-rings (i.e. every matrix over the ring can be reduced to triangular form), and $D$-rings (i.e. every matrix over the ring can be reduced to diagonal form). Note that $D$-ring$\to$$T$-ring$\to$$F$-ring; the first implication is clear, and the second is due to Kaplansky [ibid.]. This paper generalizes Kaplansky's and Helmer's results, and obtains characterizations for $T$-rings and $D$-rings.

The second paper is mainly concerned with $F$-, $T$-, and $D$-rings which are of the form $C(X)$ (where $C(X)$ denotes the ring of all real-valued continuous functions on a completely regular Hausdorff space $X$); the corresponding spaces $X$ are called, respectively, $F$-spaces, $T$-spaces, and $D$-spaces. Just as above, $D$-space$\to$$T$-space$\to$$F$-space, and it shows that $P$-spaces (i.e. every $G_\delta$ is open) and extremally disconnected spaces (i.e. the closure of an open set is open) are $D$-spaces. Examples show that none of these implications are reversible (for instance, there exists a compact, connected $D$-space); this implies the new algebraic result that the implications in the previous paragraph are also not reversible. All these spaces are somewhat unusual, since a metrizable $F$-space is shown to be discrete; non-trivial $F$-spaces can be obtained by taking $\beta X-X$ for any locally compact, $\sigma$-compact $X$ (where $\beta X$ is the Stone-Čech compactification of $X$).

In conclusion, here are some characterizations obtained in the second paper. (1) $X$ is an $F$-space (resp. $T$-space) if and only if $\beta X$ is. (For $D$-spaces, this is an open question.) (2) $X$ is an $F$-space$\leftrightarrow$ for all $f\in C(X)$ there exists a $g\in C(X)$ such that $g(X)=0$ whenever $f(x)>0$ and $g(x)=1$ whenever $f(x)<0$$\leftrightarrow$for every maximal ideal $M$ of $C(X)$, the intersection of the prime ideals contained in $M$ is prime.

*E. A. Michael* (Princeton, N.J.).

**Eilenberg, Samuel; Nagao, Hirosi; and Nakayama, Tadasi. On the dimension of modules and algebras. IV. Dimension of residue rings of hereditary rings.** Nagoya Math. J. **10** (1956), 87–95.

[For the basic notions and the terminology, see Cartan and Eilenberg, Homological algebra; Princeton, 1956;

MR 17, 1040. Some additional earlier results used here can be found in part III, by M. Auslander, Nagoya Math. J. **9** (1955), 67–77; MR 17, 579; and in a paper by Eilenberg, Comment. Math. Helv. **28** (1954), 310–319; MR 16, 442.]

The first main result proved here is: let $R$ be a hereditary ring (i.e., $\mathrm{l.gl.dim}(R) \leq 1$, or, equivalently, all left ideals of $R$ are projective $R$-modules) and let $I$ be a two sided ideal of $R$ such that $I^n = I^{n+1}$ for some $n > 0$. Then $\mathrm{l.gl.dim}(R/I) \leq 2n-1$. In particular, this implies that if $R$ is a hereditary ring containing a nilpotent two sided ideal $N$ (its radical) such that $R/N$ is semisimple with minimum condition then $\mathrm{l.gl.dim}(R/I) < \infty$, for every two sided ideal $I$, because in this case the sequence of powers $I^n$ is always eventually constant. The main new device for proving this result is the following: $R/I$-projective resolution of $J/I$, where $J$ is any left ideal of $R$ containing $I$:

$$\cdots \to I^n J/I^{n+1} J \to I^n/I^{n+1} \to I^{n-1} J/I^n J \to \cdots \to J/IJ \to J/I \to 0.$$

This also yields more precise results on the global dimension of $R/I$, of which we mention only the following: if $R$ is hereditary and has the nilpotent radical $N$ (in the sense used above, i.e., such that $R/N$ is semisimple with minimum condition) then $\mathrm{l.gl.dim}(R/N^2) = n$, where $n$ is the index of nilpotency of $N$, i.e., $N^{n+1} = 0$ and $N^n \neq 0$.

It is then shown by examples that the main result is best possible. Thus, for each pair $(m, n)$, with $0 < m \leq \infty$ and $0 < n \leq \infty$, but $(m, n) \neq (1, \infty)$, there is given a ring $R$ with nilpotent radical, and a two sided ideal $I$ of $R$, such that $\mathrm{l.gl.dim}(R) = m$ and $\mathrm{l.gl.dim}(R/I) = n$.

Finally, all these results are carried over to the cohomological dimension of algebras finite over a commutative field. This is done quite directly, using the known result that if such an algebra $R$ has finite cohomological dimension then $R/N$ is separable, which carries over to every residue class algebra of $R$, and that the separability of $R/N$ entails the coincidence of the cohomological dimension of $R$ with its global dimension.

*G. P. Hochschild* (Berkeley, Calif.).

**Evans, Trevor. Some remarks on a paper by R. H. Bruck.** Proc. Amer. Math. Soc. 7 (1956), 211–220.

The author generalizes some of the results of R. H. Bruck and shows that there are uncountable many nonisomorphic associative right neorings with a two-sided identity which generates the additive loop; moreover, the construction yields all right neorings with this property. The question of universal right neorings raised by Bruck is answered for some classes of right neorings.

The relationship between neorings and logarithmetics is discussed and the following theorem is obtained. If $Q$ is a loop, the logarithmetic $L(Q)$ of $Q$ is a left neoring with an identity which generates its additive loop. *L. J. Paige.*

**Hochschild, G. Note on Lie algebra kernels in characteristic $p$.** Proc. Amer. Math. Soc. 7 (1956), 551–557.

Let $L$ be a Lie algebra over a field $F$. An $L$-kernel is the object formed by a Lie algebra $K$ over $F$ and a homomorphism $\varphi$ of $L$ into $D(K)/I(K)$, where $D(K)$ is the derivation algebra of $K$ and $I(K)$ the ideal of inner derivations. The center $C$ of $K$ is called the center of the kernel; it has a structure of $L$-module. An $L$-kernel $(K, \varphi)$ is called extendible if $\varphi$ may be obtained in the natural manner from an extension of $K$ by $L$. There is defined an additive law of composition between $L$-kernels with a given center $C$; the relation "$(K, \varphi)$ and $(K', \varphi')$ differ from each other by an extendible kernel" is an equivalence relation in the class of $L$-kernels of center $C$, compatible with the law of composition in this set. The set of equivalence classes relative to this relation may be identified with a subgroup of the 3-dimensional cohomology group $H^3(L, C)$; the element of $H^3(L, C)$ which corresponds to a given $L$-kernel is called the obstruction of this kernel. An element $u$ of $H^3(L, C)$ is called effaceable if there is a finite dimensional $L$-module $C'$ containing $C$ such that the canonical image of $u$ in $H^3(L, C')$ is 0. The object of the present paper is to complete the proof of the following theorem: An element of $H^3(L, C)$ is the obstruction of an $L$-kernel of center $C$ if and only if it is effaceable. The "if" part and the characteristic 0 case of the "only if" part have already been established by the author [Amer. J. Math. **76** (1954), 698–716; 763–778; MR **16**, 109, 443]. Thus, what remained to be done is the "only if" part in case of characteristic $p > 0$. This is accomplished by associating to an $L$-kernel $(K, \varphi)$ a pair $(L^*, (K^*, \varphi^*))$, where $L^*$ is a restricted Lie algebra containing $L$, $K^*$ a restricted Lie algebra containing $K$ and $\varphi^*$ a restricted homomorphism of $L^*$ into $D_r(K^*)/I(K^*)$, where $D_r(K^*)$ is the algebra of restricted derivations of $K^*$; $(K^*, \varphi^*)$ may then be called a restricted $L^*$-kernel. Its obstruction is easily seen to be effaceable because the restricted universal enveloping algebra of a restricted Lie algebra is finite dimensional. From this, the author deduces that the obstruction of the original $L$-kernel was effaceable.

*C. Chevalley* (Paris).

**Mrówka, S. On the ideals' extension theorem and its equivalence to the axiom of choice.** Fund. Math. **43** (1956), 46–49.

If $A$ is a subset of a Boolean algebra $B$, and if $M$ is the set of all ideals of $B$ disjoint from $A$, then every element of $M$ is included in a maximal element of $M$. This fact is a consequence of the known maximal ideal theorem and, hence, of the axiom of choice. The contribution of the present paper is to prove that, conversely, this generalized maximal ideal theorem implies the axiom of choice.

*P. R. Halmos* (Chicago, Ill.).

**Bell, C. B. On the structure of algebras and homomorphisms.** Proc. Amer. Math. Soc. 7 (1956), 483–492.

The algebras here discussed are Boolean algebras of sets. The main result of the paper can be stated as follows. Suppose that $K$ and $L$ are classes of sets and that $f$ is a mapping from $K$ onto $L$. Suppose that whenever $\{A_v\}$ is a family of cardinality $m$ in $K$ such that $\bigcap_v A_v^{\pm 1} = \emptyset$ (the ambiguous signs indicate that, for each $v$, either $A_v$ or its complement is used), then $\bigcap_v (f(A_v))^{\pm 1} = \emptyset$ (with, of course, the same distribution of signs). Conclusion: $f$ can be extended to an $m$-homomorphism from the $m$-complete Boolean algebra generated by $K$ onto the $m$-complete Boolean algebra generated by $L$. *P. R. Halmos.*

**Kustaanheimo, Paul. On the equivalence of some calculi of transformable quantities.** Soc. Sci. Fenn. Comment. Phys.-Math. **17** (1955), no. 9, 35 pp.

A number of calculi used in operating with geometrical or physical quantities partly cover one another. In this paper an elementary mapping of many of these calculi onto each other is given by means of a "bridge algebra" that contains them all and which is called the algebra of tensor rings. It is essential that this tensor ring contains besides addition and multiplication a third operation,

called "conjugation", that is an involutory antiautomorphism. After nine chapters in which the "bridge algebra" is developed and which contain many examples, the last five chapters give the application to Ricci-Calculus, Riemannian geometry, matrices, Cracovians, Dirac's formalism, the dynamics of Gibbs and the multilinear vector functions of Nevanlinna. Many references are given. *J. A. Schouten* (Epe).

**Fujiwara, Tsuyoshi. On the existence of algebraically closed algebraic extensions.** Osaka Math. J. 8 (1956), 23–33.

The author continues the investigations by Shoda [same J. 4 (1952), 133–143; MR 14, 614] on general algebraic systems, in particular the concept of algebraic extensions. In the present paper various results are derived which make it possible to transform the condition of Shoda for algebraically closed extensions into another form. *O. Ore* (New Haven, Conn.).

**Krull, Wolfgang. Charakterentopologie, Isomorphismentopologie und Bewertungstopologie.** Mem. Mat. Inst. "Jorge Juan" no. 16 (1955), i+74 pp.

The present paper is concerned with the study, from a very general point of view, of a method for the introduction of topological notions in the study of algebraic objects. The basic notion is that of a character-topology, which is defined as follows. It is assumed that a structure $K^*$ of a certain type (not precised) is given, together with a system of substructures $K_\rho$ which, taken together, generate $K^*$. On the other hand, another structure $A^*$ (which may or may not be of the same type as $K^*$) is given; for each $K_\rho$ (and for $K^*$) itself there is given a class of mappings of $K_\rho$ (or $K^*$) into $A^*$, the class of characters, which satisfy some very simple conditions. Then, using the same procedure as for Galois groups of infinite algebraic extensions, one may define a topology and a uniform structure on the set $P$ of characters of $K^*$. The space $P$ is always complete. Besides compactness, the author introduces for $P$ the weaker notion of $U$-filter compactness, which seems better adapted to algebraic uses.

In particular, one may consider the case where $A^*=K^*$, the characters being isomorphisms of substructures of $K^*$ to other substructures of $K^*$, one then arrives at generalizations of Galois theory. The group $G$ of automorphisms of $K^*$ is a subset of $P$; this subset is not closed in general, which shows that $G$ need not be complete. More particularly, the author considers the case where $K^*$ is a field and where the characters are submitted to the condition of leaving fixed the elements of some subfield $K_0$. An example is constructed of an extension $K^*/K_0$ of transcendence degree one such that, $L$ being the field of elements of $K^*$ algebraic over $K_0$, there is an automorphism $\sigma$ of $L/K_0$ which is not extendable to an isomorphism of $K^*$ with a subfield of itself, although it is the limit of a sequence of automorphisms which are so extendable.

Another case which is considered by the author is the one where $K^*$ is a group $G$ and $P$ a set of "valuations" of $G$, i.e. of homomorphisms of $G$ into the real numbers; this case occurs for instance in the arithmetic of infinite algebraic number fields. A subset $M$ of $G$ is called a "module" when the condition "for any $w \in P$ there is an $a' \in M$ such that $w(a') \leq w(a)$ "implies" $a \in M$". If it is furthermore assumed that, for any $w \in P$, the numbers $w(a)$ $(a \in M)$ are bounded from below, then $w \to \inf_{a \in M} w(a)$ is a function on $P$, which is called an ideal function; the author studies such functions, and, in particular, the problem of characterizing them by topological conditions. This problem is closely related to that of constructing, under general conditions, a multiplicative ideal theory which preserves some of the features of the Dedekind ideal theory in finite fields. The author concludes that, although such attempts are not unsuccessful, they yield very little information on the basic algebraic objects at hand (for instance the structure of the group of elements $\neq 0$ in an infinite field), because the notions one is working with are so heavily loaded with topology that they keep only very loose connexions with the basic algebraic objects. *C. Chevalley* (Paris).

**See also**: Mrówka, p. 10; Pierce, p. 6; Taussky, p. 11; Boyer, p. 12; Molčanov, p. 12; Pyateckiĭ-Šapiro, p. 19; Hasse, p. 20; Curtis, p. 60; Adams, p. 59.

## *Groups, Generalized Groups*

**Newman, Morris. The normalizer of certain modular subgroups.** Canad. J. Math. 8 (1956), 29–31.

Let $G$ be the modular group (of all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with integers $a, b, c, d$ such that $ad-bc=1$) and $G_0(n)$ its subgroup characterized by $c\equiv 0 \pmod n$, where $n$ is a positive integer. Writing $n=2^\alpha 3^\beta n_0$, where $(n_0, 6)=1$, the author proves that the normalizer of $G_0(n)$ in $G$ is $G_0(n/2^u 3^v)$, where

$$u=\min(3, [\tfrac{1}{2}\alpha]), \quad v=\min(1, [\tfrac{1}{2}\beta]).$$

{The author's proof of this elementary theorem can be considerably simplified.} *H. D. Kloosterman* (Leiden).

**Newman, Morris. An alternative proof of a theorem on unimodular groups.** Proc. Amer. Math. Soc. 6 (1955), 998–1000.

The theorem in question is: Let $G$ be the full modular group (of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with integers $a, b, c, d$ such that $ad-bc=1$) and $G_0(n)$ its subgroup characterized by $c\equiv 0 \pmod n$, where $n$ is a positive integer. Then any subgroup $H$ of $G$ containing $G_0(n)$ is of type $H=G_0(m)$, where $m|n$. *H. D. Kloosterman* (Leiden).

**Neumann, B. H. On a question of Gaschütz.** Arch. Math. 7 (1956), 87–90.

Let $G$ be an $n$-generator group, and $\Gamma$ the set of the ordered $n$-tuples $(g_1, \cdots, g_n)$ which generate it. An automorphism of $G$ induces a permutation of $\Gamma$, and so does an automorphism of the free group with $n$ generators. The permutations so induced generate a permutation group $P$, and the systems of transitivity of $P$ are called $T$-systems of $G$. If $G$ is abelian, it has, for each relevant $n$, only one $T$-system, but an example is here given of a nilpotent group, of order $2^{15}$, with two $T$-systems for the case $n=2$. The chief weapon is the fact that in this case the order of the commutator $g_1^{-1}g_2^{-1}g_1g_2$ depends only on the $T$-system to which the pair $(g_1, g_2)$ belongs. *Graham Higman* (Oxford).

**Neumann, Hanna. On the intersection of finitely generated free groups.** Publ. Math. Debrecen 4 (1956), 186–189.

A. G. Howson [J. London Math. Soc. 29 (1954), 428–

434; MR 16, 444] showed that the intersection of two finitely generated subgroups $U$ and $V$ of a free group is finitely generated, giving the bound $N \leq 2mn-m-n+1$ for the rank $N$ of $U \cap V$ in terms of the ranks $m$ and $n$ of $U$ and $V$. The present author shows that Howson's argument, at the basis of which is the Dehn group diagram, may be modified to yield the improved bound $N \leq 2mn-2m-n+1$, if $m \geq n$, and neither $U$ nor $V$ is cyclic. Howson gave an example with $N=mn-m+2$; the author conjectures $N \leq mn-m-n+2$.

*D. G. Higman* (Ann Arbor, Mich.).

**Bognár, M. Ein einfaches Beispiel direkt unzerlegbarer abelscher Gruppen.** Publ. Math. Debrecen 4 (1956), 509–511.

The serving (or pure) subgroups of the additive group of $p$-adic integers provide examples of indecomposable groups of arbitrary finite or infinite rank not exceeding $c$, the power of the continuum. However, no explicit construction of these indecomposable groups is known. It is the purpose of this paper to present explicit constructions of indecomposable groups of any finite or infinite rank not exceeding $c$.

Let $R$ be the direct sum of groups $R_\alpha$, where each $R_\alpha$ is isomorphic to the additive group of rational numbers, and $\alpha$ ranges over an index set $K$ of power $\leq c$.

For each $\alpha \in K$, let $M_\alpha$ be a set of prime numbers with the properties: a) The prime 2 belongs to no $M_\alpha$; b) for each pair $\alpha$, $\beta$ ($\alpha \neq \beta$) in $K$, there exists a prime number $q$ which lies in $M_\alpha$ but not in $M_\beta$.

Now choose a fixed non-zero element $e_\alpha$ from each $R_\alpha$, and let $G$ be the subgroup of $R$ generated by the elements $e_\alpha/q^m$ and $\frac{1}{2}(e_\alpha+e_\beta)$, where $q$ ranges over $M_\alpha$, $m$ ranges over the non-negative integers, $\alpha$ and $\beta$ range over $K$. Then, the author shows that $G$ (of rank equal to the power of $K$) is indecomposable.

{To the reviewer's knowledge, it is still an open question whether there exist indecomposable abelian torsion-free groups of an arbitrary infinite cardinal number.}

*C. C. Faith* (East Lansing, Mich.).

**Szele, T. On quasi-indecomposable abelian torsion groups.** Acta Math. Acad. Sci. Hungar. 7 (1956), 109–114. (Russian summary)

(This is a posthumous paper which was edited by L. Fuchs). Call a primary abelian group quasi-indecomposable if it cannot be written as a direct sum of an infinite number of groups. It is shown that such a group has power at most the continuum. An example is the torsion subgroup $A$ of the complete direct sum of cyclic groups of order $p^n$ ($n=1, 2, \cdots$). It is also shown that all direct summands of $A$ are finite or have the power of the continuum; and that the endomorphic images of $A$ are finite, countable, or have the power of the continuum.

*I. Kaplansky* (Princeton, N.J.).

**Fuchs, L. On abelian torsion groups which can not be represented as the direct sum of a given cardinal number of components.** Acta Math. Acad. Sci. Hungar. 7 (1956), 115–124. (Russian summary)

The author generalizes the results of the preceding paper. For a cardinal number $m$, call a primary abelian group $m$-indecomposable if it admits no decomposition into $m$ direct summands. Szele's paper concerns the case $m=\aleph_0$. Again $t=m^{\aleph_0}$ is a bound for the power of an $m$-indecomposable group. Set-theoretic conditions on $m$ are given which are equivalent to the existence of an $m$-indecomposable group. If one does exist, the number of such is equal to the cardinal number $2^t$ of all groups of power $t$.

*I. Kaplansky* (Princeton, N.J.).

**Boyer, D. L. Enumeration theorems in infinite Abelian groups.** Proc. Amer. Math. Soc. 7 (1956), 565–570.

Theorem 1. Determination of the number of subgroups of a countable abelian group: the answer is $\aleph_0$ or continuum, depending on a certain criterion. Theorem 2. Extension to modules over a principal ideal ring. Theorem 3. Any countable torsion abelian group has continuum many automorphisms.

*I. Kaplansky.*

**Neumann, B. H. On some finite groups with trivial multiplicator.** Publ. Math. Debrecen 4 (1956), 190–194.

If a finite group $G$ is generated by $d$ elements and defined by $e$ relations between these generators, then $e \leq d$. It is known that if $e=d$ the Schur multiplicator of $G$ is trivial [J. Reine Angew. Math. 127 (1904), 20–50; 132 (1907), 85–137]. The author raises the question whether every finite group with trivial multiplicator (abgeschlossen in Schur's sense) can be so presented that $e=d$. He shows for several types of metacyclic groups that this is true. The groups considered are

(1) $\{a, b; a^m=1, b^n=1, b^{-1}ab=a^r\}$,

where $r^n=1 \pmod m$ and $(m, n)=(m, r-1)=1$, the generalized quaternion groups

(2) $\{a, b; a^{2^m}=1, b^2=a^{2^{m-1}}, b^{-1}ab=a^{-1}\}$, $m \geq 2$,

(3) $\{a, b; a^{2^m}=1, b^2=1, b^{-1}ab=a^{-1+2^{m-1}}\}$, $m \geq 3$,

(4) $\{a, b; a^{p^m}=1, b^{p^m}=1, b^{-1}ab=a^{1+p^{m-n}}\}$,

where $m>n>0$ if $p$ is an odd prime and $m-1>n>0$ if $p=2$. These were all shown by Schur to have trivial multiplicator and each of them is shown here to be definable by 2 relations between the 2 generators $a$ and $b$.

*D. C. Murdoch* (Vancouver, B.C.).

**Rédei, Ladislaus. Die endlichen einstufig nichtnilpotenten Gruppen.** Publ. Math. Debrecen 4 (1956), 303–324.

A finite group is said to be einstufig non-nilpotent if it is non-nilpotent but all its proper subgroups are nilpotent. Similarly it is einstufig non-abelian if it is non-abelian but has only abelian proper subgroups. It is known that the order of an einstufig non-nilpotent group is divisible by exactly two primes $p$ and $q$, and that the commutator subgroup $G'$ is the only normal Sylow subgroup. If the notation is chosen so that $G'$ is a $p$-Sylow group, then the $q$-Sylow groups are cyclic and the order of $G'/G''$ is $p^u$ where $u$ is the exponent to which $j$ belongs mod $q$. Finally $G'''=1$ and $G$ is einstufig non-abelian if and only if $G''=1$. If $q^v$ is the order of the $q$-Sylow group of $G$, and therefore of $G/G'$, the numbers $p$ and $q^v$ are called the invariants of the einstufig non-nilpotent group $G$. To every pair of invariants $p$, $q^v$ there corresponds a unique einstufig non-abelian group $G$, whose order is $p^uq^v$. At least one additional einstufig non-nilpotent group with invariants $p$, $q^v$ exists if and only if $u$ is even. Among these additional ones, when $u=2t$, there is exactly one of maximal order $p^{u+t}q^v=p^{3t}q^v$. Generating relations are given for these different groups with invariants $p$ and $q^v$.

*D. C. Murdoch* (Vancouver, B.C.).

**Molčanov, A. M. Group rings of finite groups.** Dokl. Akad. Nauk SSSR (N.S.) 107 (1956), 19–21. (Russian)

Let $G$ be a finite group. For any integer $N>0$ the

author associates with each element $\alpha=(g_1, \cdots, g_N)$ of the product group $G^N=G\times\cdots\times G$ the function $p$ on $G$ such that $p(g)=N^{-1}$ times the number of indices $i$ for which $g_i=g$. If $p$, $q$, $r$ are the functions associated with $\alpha$, $\beta$, $\gamma$ respectively, and if $\gamma=\alpha\beta$, then $r$ and the convolution of $p$ and $q$ are not in general equal, but they are, in a certain sense, asymptotically equal for $N\to\infty$. This permits the author to consider the elements of the group ring (i.e. group algebra over the field of real numbers) $A$ of $G$, having coefficients $\geqq 0$ and trace 1, as being limits as $N\to\infty$ of certain equivalence classes of $G^N$. Further considerations lead to the elements of $A$ with arbitrary real coefficients of absolute value sum $\leqq 1$, and even to the analogous elements of the group algebra over the field of complex numbers. *E. Kolchin.*

**Zappa, Guido. Sui gruppi finiti risolubili per cui il reticolo dei sottogruppi di composizione è distributivo.** Boll. Un. Mat. Ital. (3) **11** (1956), 150–157.

The main result of this paper is the following theorem. A necessary and sufficient condition that the lattice of composition subgroups (groups appearing in a composition series) in a finite solvable group be distributive is that all Sylow subgroups of the group be cyclic. *O. Ore.*

**Higman, Graham. Complementation of abelian normal subgroups.** Publ. Math. Debrecen **4** (1956), 455–458.

The main result of this paper is the following. Let $N$ be an abelian normal subgroup of a finite group $G$. Suppose that for each prime $p$ there exists a subgroup $X$ of order prime to $p$ such that $NX$ is normal in $G$, but not $N_0X$ for any normal subgroup $N_0$ of $G$ such that $N_0$ is a subgroup of $p$-power index $\neq 1$ in $N$. Then, if $H$ is any group containing $N$ and $G$ as normal subgroups (e.g., $H=G$), there is a complement for $N$ in $H$ and any two complements are conjugate. The case in which $H=G$ and $N$ is a $p$-group is established first, a crucial point being an application of a theorem of Schur implying conjugacy of complements when $N$ and $G/N$ have relatively prime orders [Zassenhaus, Lehrbuch der Gruppentheorie, Bd. 1, Teubner, Leipzig-Berlin, 1937, p. 125]. The reduction of the general result to this case involves results of W. Gaschütz [J. Reine Angew. Math. **190** (1952), 93–107; MR **14**, 445] and the reviewer [Pacific J. Math. **4** (1954), 545–555; MR **16**, 565]. *D. G. Higman* (Ann Arbor, Mich.).

**Žmud', È. M. On isomorphic linear representations of finite groups.** Mat. Sb. N.S. **38(80)** (1956), 417–430. (Russian)

Let $\mathfrak{G}$ be a finite group and $k$ a given positive integer. The author gives two necessary and sufficient conditions for $\mathfrak{G}$ to admit a faithful representation containing exactly $k$ absolutely irreducible components (the ground field of all the representations is assumed to be of characteristic zero). Let $\mathfrak{M}$ be the sockel of $\mathfrak{G}$, i.e. the group generated by the minimal invariant subgroups of $\mathfrak{G}$. Consider $\mathfrak{M}$ as an operator group with respect to $\mathfrak{G}$. Then the first condition requires that $\mathfrak{M}$ be generated by $k$ elements. [For $k=1$, see Gaschütz, Math. Nachr. **12** (1954), 253–255; MR **16**, 671]. — To state the second condition, consider the minimal characteristic subgroups of $\mathfrak{M}$. (By construction, $\mathfrak{M}$ is the direct product of its minimal characteristic subgroups.) Let $\mathfrak{K}$ be one of them. Then $\mathfrak{K}$ contains a minimal subgroup $\mathfrak{F}$; the latter is uniquely determined up to isomorphisms. If $\mathfrak{F}$ is non-abelian then $\mathfrak{K}=\mathfrak{F}$. Consider the case where $\mathfrak{F}$ is abelian. Then $\mathfrak{F}$ is $p$-elementary for some prime number $p$. There are three numerical invariants belonging to $\mathfrak{K}$ and $\mathfrak{F}$: (1) The order $p^r$ of $\mathfrak{F}$. (2) The order of $\mathfrak{K}$ which is $p^{rs}$ with some positive integer $s$. (3) The number $p^g$ of elements in the field $K$ of endomorphisms of $\mathfrak{F}$. With these notations, the second condition may be stated as follows: For each non-abelian $\mathfrak{K}$ the inequality $k\geqq sg/r$ holds. [For $k=1$, see Shoda, J. Fac. Sci. Imp. Univ. Tokyo. Sect. I. **2** (1931), 203–209; for arbitrary $k$ see Tazawa, Tôhoku Math. J. **47** (1940), 87–93; MR **2**, 3; see also Kochendörffer, Math. Nachr. **1** (1948), 25–39; MR **10**, 281.] — The proofs in this beautiful paper are obtained by computing the number $\varrho_k(\mathfrak{G})=\sum_\Delta n(\Delta)^2$ where $\Delta$ ranges over the faithful irreducible representations of $\mathfrak{G}$ which contain exactly $k$ absolutely irreducible components, and where $n(\Delta)$ denotes the product of the degrees of the components contained in $\Delta$. The author shows first that $\varrho_k(\mathfrak{G})=(\mathfrak{G}:\mathfrak{M})^k\sum_T\mu(T)(\mathfrak{M}:\mathfrak{H}_T)^k$ where $T$ ranges over the sets of minimal invariant subgroups of $\mathfrak{G}$ (including the empty set); $\mu(T)=(-1)^t$ where $t$ is the number of elements in the set $T$; $\mathfrak{H}_T$ denotes the group generated by the groups of the set $T$. There is an anti-automorphism $\mathfrak{H}\to\overline{\mathfrak{H}}$ of the lattice of subgroups of $\mathfrak{H}$ of $\mathfrak{M}$, and therefore $\varrho_k(\mathfrak{G})=(\mathfrak{G}:\mathfrak{M})^k\sum_T\mu(\overline{T})(\mathfrak{M}:\overline{\mathfrak{H}}_T)^k$. From this, the author deduces the first condition. In order to prove the second condition, he shows that

$$\varrho_k(\mathfrak{A})=(\mathfrak{G}:\mathfrak{M})^k\prod^{\mathfrak{N}}(N-1)\prod^{\mathfrak{K}}\prod_{0\leqq\nu<s}(p^{kr}-p^{\nu g})$$

where $\mathfrak{N}$ ranges over the non-abelian minimal characteristic subgroups of $\mathfrak{M}$ of order $N$, and $\mathfrak{K}$ ranges over the abelian ones with $r$, $s$, $g$, $p$ defined as above.

*P. Roquette* (Hamburg).

**Robinson, G. de B. The degree of an irreducible representation of $S_n$.** Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 357–359.

W. Feit proved by induction [Proc. Amer. Math. Soc. **4** (1953), 740–744; MR **15**, 287] the formula $f_\lambda=n!|z_{ij}|$ for the degree $f_\lambda$ of the irreducible representation of $S_n$ corresponding to the Young diagram $[\lambda]$, where $z_{ij}=1/(\lambda_j-j+i)!$. The ratio $n!/f_\lambda$ was shown by Frame, Robinson, and Thrall [Canad. J. Math. **6** (1954), 316–324; MR **15**, 931] to be the product of the hook lengths $h_{ij}=(\lambda_i-i)+(\lambda_j'-j)+1$ in $[\lambda]$. The author now obtains a direct proof of Feit's formula without using induction by comparing directly the degrees of the so called determinantal terms which remain uncancelled after expanding the Robinson and Taulbee formula

$$[\lambda]=\prod(1-R_{ij})[\lambda_1]\cdot[\lambda_2]\cdots[\lambda_n]$$

[Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 723–726; MR **16**, 110] with the terms in Feit's determinant. *J. S. Frame.*

**Schützenberger, Marcel Paul. Sur une représentation des demi-groupes.** C. R. Acad. Sci. Paris **242** (1956), 2907–2908.

This paper depends on a previous one [same C. R. **242** (1956), 862–864; MR **17**, 702] in the course of which a certain representation of a "fundamental semi-group of a regenerative process" by square matrices occurs: it is called a right-ergodic representation. It would be difficult to summarize the present short paper without repeating the definitions and notations in full, but the last of its four propositions gives conditions under which the right-ergodic representation is an isomorphism of the semi-group or of a certain quotient semi-group.

*H. A. Thurston* (Bristol).

**Kemperman, J. H. B. On complexes in a semigroup.** Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 247–254.

Let $G$ be a semigroup written additively, i.e. a set with associative addition and left and right cancellation laws. $A$, $B$ denote finite non-empty subsets of $G$, and $a$, $b$ denote arbitrary elements of $A$, $B$ respectively. $A+B$ is the set of all sums $a+b$. The number of elements in $A$ is denoted $[A]$, and $k(A, B)$ stands for $[A]+[B]-[A+B]$. An element $c$ is called invertible if $G$ can be embedded in a semigroup $G'$ containing an identity 0 and an inverse $-c$ of $c$.

The main result (Theorem 1) (stated to have been found independently for groups by D. F. Wehn) asserts that each invertible element in $A+B$ admits at least $k(A, B)$ representations in the form $a+b$. A generalization (Theorem 4) is that if $A_1, \cdots, A_n$ are $n$ ($\geqq 2$) finite non-empty subsets of $G$, each invertible element in $C=A_1+\cdots+A_n$ admits at least $\sum[A_i]-[C]-n+2$ representations as $a_1+\cdots+a_n$ ($a_i \in A_i$).

Theorem 1 is significant only if $k(A, B)\geqq 2$, in which case it is stated that $G$ contains elements of finite order. {This seems to be an exaggeration. What is proved is that this is so if there is an element of $A+B$ possessing an inverse in $G$.} This and other considerations which apply when $G$ is a group lead to the following conjecture: if $G$ is a group in which 0 is the only element of finite order, and $[B]\geqq 2$, then at least 2 elements of $A+B$ admit only 1 representation as $a+b$. The conjecture is verified if $G$ is group-isomorphic with an ordered group, if $G$ is abelian, or if merely all pairs of elements of $B$ commute.

*I. M. H. Etherington* (Edinburgh).

**Cohn, P. M. Embeddings in sesquilateral division semigroups.** J. London Math. Soc. **31** (1956), 181–191.

A semigroup $S$ is called a Dr-semigroup if to every couple $a, b \in S$ there is $ax \in S$ with $xa=b$. $S$ is called an SDl-semigroup if for any two $a, b \in S$ either $a=b$ or $a=bz$ or $b=az$ for some $z \in S$. It is pointed out that a semigroup satisfying Dr and SDl need not be a group.

1. An SDl-semigroup having an idempotent is embeddable in a semigroup satisfying Dr and SDl if and only if $S$ is embeddable in a group. 2. An SDl-semigroup $S$ without idempotents is embeddable in a semigroup satisfying Dr and SDl if and only if in $S$ the left cancellation law holds.

*Št. Schwarz* (Bratislava).

**Cohn, P. M. Embeddings in semigroups with one-sided division.** J. London Math. Soc. **31** (1956), 169–181.

A semigroups $S$ is called a Dr-semigroup if to every couple $a, b \in S$ there exists an $x \in S$ with $xa=b$. (For such semigroups the term "left simple semigroups" is now often used.) A left cancellation semigroup is called a Cl-semigroup. Necessary and sufficient conditions for a semigroup to be embeddable in a Dr-semigroup are given.

Main results: 1. Suppose that $S$ contains no idempotent. Then $S$ is embeddable in a Dr-semigroup if and only if for all $x, y, u, v \in S$, $ux=uy$ implies $vx=vy$. 2. Suppose that $S$ has at least one idempotent $e$. Then $S$ is embeddable in a Dr-semigroup if and only if a) for all $x, y, u, v \in S$, $ux=uy$ implies $vx=vy$, b) $eS$ is embeddable in a group, c) if in a chain $a_0, a_1, \cdots, a_n \in S$ we have $a_{i-1}S\cap a_iS\neq\emptyset$ ($i=1, 2, \cdots, n$) and $ea_0=ea_n$, then $a_0=a_n$. (There is an example showing that the condition c) cannot be omitted.) 3. A Cl-semigroup without idempotents is always embeddable in a Dr-semigroup. 4. A Cl-semigroup $S$ having an idempotent is embeddable in a Dr-semigroup if and only if $S$ is embeddable in a group.

The by far most difficult proof is that of the sufficiency of the statement 1.

*Št. Schwarz* (Bratislava).

**Wallace, A. D. The Gebietstreue in semigroups.** Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 271–274.

Let $S$ be a topological semigroup, and for $t \in S$, let $\Gamma(t)=\{t^n: n\geqq 1\}^-$ (the closure of the powers of $t$). If $A\subset S$, let $P(A)=\{t \in S: tA=A\}$. $A$ will be called an $(n, G)$-rim for $S$ if the natural homomorphism $H^n(S; G)\to H^n(B, G)$ is not onto for any closed proper subset $B$ of $S$ that contains $A$. Here $G$ is any abelian group, and $H$ denotes the cohomology group. If $S$ is compact then $K$ will denote the minimal ideal. $S$ is simple if $K=S$ or equivalently, if $S\times S=S$ for each $x \in S$.

Theorem. Let $S$ be compact and let $A$ be a closed $(n, G)$-rim for $S$. Then $P(A)\subset P(S)$. If in addition $S$ is connected and not simple and $P(A)$ is non-null, then $AS=S=K\cup SA$, and if $P(S)=P(A)$, then $P(S)\subset A$. Corollary. Let $S$ be compact, connected, contained in Euclidean $n$-space ($n\geqq 2$), and let $A$ be the boundary of $S$. If $P(A)$ is non-null, and $S\neq K$, then $P(A)=P(S)\subset A$. This includes the known result [Wallace, Math. J. Okayama Univ. 3 (1953), 1–3; MR 15, 933] that if $S$ is a compact connected semigroup with identity in $E_n$, then all elements with inverses are contained in the boundary.

Let $X$ be compact Hausdorff and let $M(X)$ be all continuous maps of $X$ into $X$. $M(X)$ is a topological semigroup if we use the compact open topology and take composition of functions as the multiplication. Theorem. If $A$ is a closed $(n, G)$-rim for $X$ and $f \in M(X)$ with $f(A)=A$, and $\Gamma(f)$ compact, then $\Gamma(f)$ acts as a group of homeomorphisms taking $X$ onto $X$.

Theorem. If $S$ is compact and is the union of groups and if the set $E$ of idempotents of $S$ is contained in some $(n, G)$-rim of $S$, then $S=E$.

*A. Shields.*

**Wallace, A. D. The Rees-Suschkewitsch structure theorem for compact simple semigroups.** Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 430–432.

The author extends the structure theorem of Suschkewitsch [Math. Ann. **99** (1928), 30–50] and Rees [Proc. Cambridge Philos. Soc. **36** (1940), 387–400; MR **2**, 127] to the case of compact simple semigroup $S$. Here $S$ is called simple if $SxS=S$ for all $x \in S$, or equivalently, $S=K$ where $K$ denotes the minimal ideal of $S$. Let $\mathfrak{L}$ be the set of minimal left ideals of $S$, $\mathfrak{R}$ the set of minimal right ideals. These are easily made into compact semigroups. If $e^2=e$, let $H(e)$ be the maximal subgroup of $S$ containing $e$ (necessarily compact). Let $X=H(e)\times\mathfrak{L}\times\mathfrak{R}$. The main result is that when $X$ is endowed with a suitable multiplication (not the cartesian product multiplication) it is topologically isomorphic to $K$. The proof is not given in detail. A number of interesting corollaries follow from this. For example, if an $n$-sphere is a simple semigroup then either it is a group (and $n=1$ or 3) or the multiplication must be one of the two trivial possibilities: (i) $xy=x$ for all $x, y$, (ii) $xy=y$ for all $x, y$.

*A. Shields.*

**Cowell, W. R. Concerning a class of permutable congruence relations on loops.** Proc. Amer. Math. Soc. **7** (1956), 583–588.

It is known [see G. Birkhoff, Lattice theory, Amer. Math. Soc. Colloq. Publ., vol 25, rev. ed., New York, 1948; MR **10**, 673] that under certain simple conditions all congruences on a loop commute, and the present

paper weakens these conditions. The author states that congruences on a loop are not in general permutable; correspondence with the author reveals that a suitable example is a certain loop isotopic to the quasigroup used by the reviewer [Proc. Amer. Math. Soc. 3 (1952), 363–366; MR 14, 241] to prove the corresponding result for quasigroups.

A congruence $\theta$ on a loop $G$ is called quasinormal if, for each $x$, the congruence-class of $x$ is $x\Delta$, where $\Delta$ is the semigroup of permutations of the elements of $G$ generated by right and left translations of elements of the kernel of $\theta$. The first theorem is that any quasi-normal congruence on $G$ commutes with any congruence on $G$. A sufficient condition for all congruences on $G$ to be quasinormal is given. Finally it is proved that conditions that a given congruence relation should commute with all others cannot be stated in terms of the quotient-loop alone.

*H. A. Thurston* (Bristol).

**Rédei, Ladislaus. Die gruppentheoretischen Zetafunktionen und der Satz von Hajós.** Acta Math. Acad. Sci. Hungar. 6 (1955), 271–279. (Russian summary)

In an earlier paper [same Acta 6 (1955), 5–25; MR 17, 344] the author introduced a zeta-function associated with subgroups of an abelian group. He now shows how the theorem of Hajós can be expressed in terms of such zeta-functions.

Let $n$ be a non-negative integer and $\mathfrak{N}$ the set of numbers $1, 2, \cdots, n$. Let $A_1, A_2, \cdots, A_n$ be cyclic subgroups of a given finite abelian group $G$, and, for any $\mathfrak{M}\subseteq\mathfrak{N}$, let $A_{\mathfrak{M}}$ denote the subgroup generated by the elements of $A_i$ for $i\in\mathfrak{M}$. Let $B_i$ be the group of $p_i$ the powers of elements of $A_i$, where $p_1, p_2, \cdots, p_n$ are prime numbers, not necessarily different. Let

$$\varrho_{\mathfrak{M}}(z)=\varrho(z; B_{i_1}A_{\mathfrak{M}}/A_{\mathfrak{M}}, \cdots, B_{i_m}A_{\mathfrak{M}}/A_{\mathfrak{M}}),$$

where $i_1, i_2, \cdots, i_m$ are the different elements of $\mathfrak{N}-\mathfrak{M}$, and

$$\varrho(z; C_1, C_2, \cdots, C_n)=\sum_{\mathfrak{M}\subseteq\mathfrak{N}}(-1)^{(\mathfrak{M})}(A_{\mathfrak{M}})^{-z}.$$

Here, for any finite set $S$, $(S)$ denotes the number of elements of $S$. In particular, when $\mathfrak{M}$ is the null set, $m=n$ and

$$\varrho_{\mathfrak{M}}(z)=\varrho_0(z)=\varrho(z; B_1, B_2, \cdots, B_n).$$

It is shown that Hajós's theorem is equivalent to the following result: If $(A_{\mathfrak{N}})=p_1p_2\cdots p_n$ and $\varrho_{\mathfrak{M}}(1)=0$ for all $\mathfrak{M}\subset\mathfrak{N}$ $(\mathfrak{M}\neq\mathfrak{N})$, then $\varrho_0(r)=0$ for $r=2, 3, 4, \cdots$. The proof is split into a series of ten lemmas and uses the author's inertia theorem proved in the paper referred to.

*R. A. Rankin* (Glasgow).

**See also:** Farahat and Mirsky, p. 4; Amitsur, p. 8; Thomas, p. 57.

# THEORY OF NUMBERS

on order

## *General Theory of Numbers*

★**Behnke, Heinrich. Vorlesungen über allgemeine Zahlentheorie. Unter Mitarbeit von Reinhold Remmert.** Ausarbeitungen mathematischer und physikalischer Vorlesungen. Bd. XVIII. Aschendorffsche Verlagsbuchhandlung, Münster, 1956. iv+180 pp. DM15.00.

This book treats elementary number theory from an algebraic standpoint, an approach not at all usual in works in English on this subject. The number theory is in effect imbedded in modern algebra, its language and its spirit. This algebraic viewpoint has perhaps influenced the author's selection of topics, since there are parts of number theory, Diophantine equations for example, that are not suited to an algebraic formulation. However, the setting aids the author in discussing number theory as part of the mainstream of mathematics, not as a subject apart or as a collection of numerical curiosities.

Chapter 1 introduces groups, rings, integral domains, fields, integers, divisibility, Zermelo's inductive proof of the unique factorization theorem, units, associated elements, ideals, norms, and multiplicative functions.

Chapter 2 treats digital questions, including the representation of real numbers by digits to a general base. Fermat's theorem is proved by periodic properties of the representation of rational numbers.

Residue classes and congruences comprise the material of Chapter 3, including another proof of Fermat's theorem from the group theoretic standpoint. The concepts of cyclic group and order of an element prepare the way for the discussion in Chapter 5 of the structure of prime power residue class groups. Primitive roots are treated as generators of the appropriate multiplicative group. The basis theorem for finite groups (every finite abelian group is the direct product of cyclic groups) is proved and applied to the residue class group structure. This leads naturally into the problem of $n$th power residues, i.e., the solutions of $x^n\equiv c \pmod m$. Akin to the direct product of groups, the notion of the direct sum of rings is employed in the decomposition of the residue class ring modulo $m$.

Chapter 5, which has perhaps less trace of the algebraic viewpoint than the others, deals with quadratic residues, the reciprocity law, Legendre's and Jacobi's symbols and the generalized reciprocity law.

The final chapter 6 treats quadratic fields, the group of units in such a field, the Pell equation, discriminants, and the factorization of a prime $p$ in quadratic fields having unique factorization.

*I. Niven.*

**Bateman, P. T.; and Erdös, P. Partitions into primes.** Publ. Math. Debrecen 4 (1956), 198–200.

The authors prove that $P(n+1)\geqq P(n)$ $(n=1, 2, 3, \cdots)$, where $P(n)$ is the number of partitions of the integer $n$ into primes $(>1)$. The proof uses only formal power series and elementary reasoning.

*J. Lehner.*

**Cheng, David K. Encoding nonintegers in a general $p$-adic number system.** Science 124 (1956), 120–121.

An elementary discussion of the representation of real numbers, and in particular, rationals, to various bases, and methods of changing from one base to another.

*L. Moser* (Edmonton, Alta.).

**Ricci, Giovanni. Aritmetica additiva: aspetti e problemi.** Confer. Sem. Mat. Univ. Bari no. 7 (1954), 31 pp. (1956).

An expository account of classical and modern results of the additive theory of numbers.

*H. B. Mann.*

**Mitrinovitch, Dragoslav S. Problème sur les progressions arithmétiques.** Boll. Un. Mat. Ital. (3) 11 (1956), 256–257.

The author gives a generalization of an elementary result of Barsotti [Soc. Parana. Mat. Annuário 1 (1954), 14–17; MR 16, 1088] to the case of two arithmetic progressions with the same number of terms and common difference.

The theorem now reads

$$\prod_{v=0}^{k} (a+vd) \equiv (-1)^{k+1} \prod_{v=0}^{k} (b+vd) \pmod{a+b+kd}.$$

{The proof is unnecessarily complicated. One needs only to note that if the order of the factors of one product is reversed, corresponding factors of both products become congruent. The product operator may be replaced by any symmetric function of the terms of the arithmetic progression.} *D. H. Lehmer* (Berkeley, Calif.).

**Kanold, Hans-Joachim. Eine Bemerkung über die Menge der vollkommenen Zahlen.** Math. Ann. 131 (1956), 390–392.

Let $N(x)$ denote the number of perfect numbers $\leqq x$. Hornfeck proved $N(x) \leqq x^{\frac{1}{2}}$ [Arch. Math. 6 (1955), 442–443; MR 17, 460]. Using a theorem of Dickson [Amer. J. Math. 35 (1913), 413–422)], the author improves this to $N(x)=o(x^{\frac{1}{2}})$. *P. Scherk* (Philadelphia, Pa.).

**Bernhard, Herbert A. On the infinitude of primitive $k$-nondeficients.** Proc. Amer. Math. Soc. 7 (1956), 469–471.

An integer $n$ is said to be $k$-deficient if $kn > \sigma(n)$, sum of divisors of $n$; otherwise $n$ is $k$-nondeficient. A primitive $k$-nondeficient integer is one that is $k$-nondeficient but whose proper divisors are $k$-deficient. H. N. Shapiro [Bull. Amer. Math. Soc. 55 (1949), 450–452; MR 10, 514] has shown that in order that there be an infinite number of primitive $k$-nondeficients with a fixed number of distinct prime factors, it is necessary that $k$ be of the form

$$\prod_{i=1}^{r} \frac{\sigma(p_i^{\alpha_i})}{p_i^{\alpha_i}} \cdot \prod_{j=1}^{s} \frac{q_j}{q_j-1},$$

where the $p_i$ and $q_j$ are distinct primes and $s \geqq 1$. The author proves that given any $k$ of the above form, there exist infinitely many primitive $k$-nondeficient numbers having $r+s+1$ or fewer distinct prime factors. {Reviewer's remark: The writer states the result in slightly stronger form which the proofs do not justify.} *I. Niven.*

**Mařík, Jan. On quadratic polynomials which take on numerous prime values.** Časopis Pěst. Mat. 78 (1953), 57–58. (Czech)

The author quotes a previous theorem [Časopis Pěst. Mat. Fys. 74 (1950), 164–165] to the effect that if $p$ is a prime for which $x^2+x+p$ is also a prime for all $x$ for which $x^2+x \leqq (p-1)/3$ then every number $<p^2$ properly represented by the form $x^2+xy+py^2$ is a prime. This theorem is closely related to a result of Frobenius [S.-B. Preuss. Akad. Wiss. 1912, 966–980; badly misquoted by Dickson in his "History of the theory of numbers", v. 1, Carnegie Inst. Washington, 1919, p. 421]. As the author points out, the known va.lues of $p$ for which the hypothesis holds are 2, 3, 5, 11, 17 and 41. Other quadratic functions representing numerous primes are listed. These include for example

$$5x^2-5x+13,\ x=1(1)12,\ 2x^2-2x+19,\ x=1(1)18,$$
$$x^2-x-109,\ x=1(1)28,\ x^2-x-169,\ x=2(1)25.$$

*D. H. Lehmer* (Berkeley, Calif.).

**Carlitz, L.; and Corson, H. H. Some special equations in a finite field.** Monatsh. Math. 60 (1956), 114–122.

Proofs are given of results announced previously [Proc. Nat. Acad. Sci. U.S.A. 41 (1955), 752–754; MR 17, 463]. In particular, necessary and sufficient conditions on the positive integers $m_1, \cdots, m_r$ are determined, under which the number of solutions of

$$a_1x_1^{m_1}+\cdots+a_rx_r^{m_r}=0,$$

in a finite field, is independent of $a_1, \cdots, a_r$ (each $a_i \neq 0$); the number of solutions being then $q^{r-1}$. {A factor $(-1)^{r-k}$ seems to be missing from the main formula of Theorem 1}. *H. Davenport* (London).

**Touchard, Jacques. Nombres exponentiels et nombres de Bernoulli.** Canad. J. Math. 8 (1956), 305–320.

Put

$$e^{e^z-1}=\sum_0^\infty \frac{a_nz^n}{n!},\quad e^{x(e^z-1)}=\sum_0^\infty \phi_n(x)\frac{z^n}{n!}.$$

A number of interesting properties of $a_n$ and $\phi_n(x)$ are proved. We shall briefly outline the principal results.

1. Using the symbolic notation $e^{az}=\sum a_nz^n/n!$ we get

$$a(a-1)\cdots(a-n+1)=1;$$

more generally if $f(x)$ is any polynomial, then

$$f(a+n)=a(a-1)\cdots(a-n+1)f(a).$$

2. It follows from Newton's formula in finite differences that

$$h_n(a)f(a)=\Delta^n f(0)+\frac{\Delta^{n+1}f(0)}{1!}+\cdots,$$

where

$$h_n(x)=\sum_{r=0}^{n}(-1)^r\binom{n}{r}x(x-1)\cdots(x-n+r+1);$$

a consequence of this result is the "symbolic orthogonality"

$$(*)\qquad h_m(a)h_n(a)=\begin{cases}0 & (m\neq n)\\ m! & (m=n).\end{cases}$$

3. Of additional formulas involving $a_n$ we cite

$$2a(2a-1)\cdots(2a-n+1)=n!\sum_{2r\leq n}\frac{1}{r!}\frac{2^{n-2r}}{(n-2r)!},$$

$$\frac{1}{(a+1)\cdots(a+n)}=\frac{1}{e}\left[e-1-\frac{1}{1!}-\cdots-\frac{1}{(n-1)!}\right].$$

Also numbers $c_n$, $q_n$ are introduced such that

$$a(a-2)(a-4)\cdots(a-2k+2)=(-1)^{k-1}c_k,$$
$$q^n=(a-a^1)^n,\ q^{n+1}=(q+1)^n-(q-1)^n.$$

It is shown that $q^2(q^2-1^2)\cdots(q^2-(k-1)^2)=\dfrac{(2k)!}{k!}$.

4. Relation of $a_n$ to the continued fraction

$$I(x)=\frac{1\,|}{|x+1}-\frac{1\,|}{|x+2}-\frac{2\,|}{|x+3}\cdots.$$

5. Relation to the entire function

$$g(s)=\frac{1}{1^s}+\frac{1}{1!2^s}+\frac{1}{2!3^s}+\cdots.$$

6, 7. Properties of $\phi_n(x)$. In particular we mention

$$\phi(\phi-1)\cdots(\phi-n+1)=x^n,$$
$$x^nf(\phi+n)=\phi(\phi-1)\cdots(\phi-n+1)f(\phi),$$

where $f(x)$ is an arbitrary polynomial. Let

$$H_n(x, z)=\sum_{r=0}^{n} (-1)^r\binom{n}{r} x^r z(z-1)\cdots(z-n+r+1),$$

then

$$\phi^k H_n(x, \phi)=0 \ (0\leq k<n),\ \phi^n H_n(x, \phi)=n!x^n.$$

8. Arithmetic properties. In particular for $p$ prime

$$A_p\equiv r+1,\ c_{p+2}-c_{p+1}\equiv c_p\equiv 1,$$
$$q_{p+1}\equiv 2,\ q_{2p}\equiv 2 \pmod{p}.$$

9, 10, 11. Definite integrals. We cite

$$\frac{\pi A_n}{\Gamma(n)}=\int_0^\infty \frac{e^{\cos z}}{(1+z^2)^{n/2}}\cos[z+e\sin z-n\arctan z]dz,$$

$$ea_n=$$
$$\int_0^\infty \frac{z^n dz}{\Gamma(1+z)}-\frac{1}{\pi}\int_0^\infty \frac{e^{-z}\Gamma(n)\sin\{n\arctan(\pi/\log z)\}}{(\pi^2+\log^2 z)^{n/2}}dz.$$

12. Exponential numbers of several variables.

$$a^n(w, w')=(wa+w'a')^n.$$

In particular $q_n=a_n(1, -1)$ and

$$e^2 q_{2k}=\sum_{n,m=0}^{\infty}\frac{(m-n)^{2k}}{m!\,n!}.$$

13. Bernoulli numbers. Define $b_n$ by means of $x/(e^x-1)=\sum_0^\infty b_n x^n/n!$. The writer constructs a sequence of polynomials $Q_n(x)$ such that

$$(**)\qquad Q_m(b)Q_n(b)=\begin{cases}0 & (m\neq n)\\ 2^n K_n & (m=m),\end{cases}$$

which may be compared (*) above; $K_n$ is given by

$$K_n=\frac{(-1)^n}{2n+1}\frac{1}{2^n}\frac{(n!)^4}{[1\cdot 3\cdots(2n-1)]^2}.$$

It is shown that $Q_{n+1}(x)=(2x+1)Q_n(x)+\dfrac{n^4}{4n^2-1}Q_{n-1}(x)$.

14. The following formula for $b_n$ is proved:

$$b_n=-\tfrac{1}{2}\pi i\int_{c-i\infty}^{c+i\infty}\frac{z^n dz}{\sin^2 \pi z}\quad(-1<c<0)$$

as a corollary of a formula of Jensen. Finally it is shown that for $-1<c<0$

$$-\tfrac{1}{2}\pi i\int_{c-i\infty}^{c+i\infty}\frac{Q_m(z)Q_n(z)}{\sin^2 \pi z}dz=\begin{cases}0 & (m\neq n)\\ 2^n K_n & (m=n)\end{cases}$$

so that the symbolic orthogonality (**) is replaced by true orthogonality. *L. Carlitz* (Durham, N.C.).

**Wyman, Max; and Moser, Leo. On some polynomials of Touchard.** Canad. J. Math. 8 (1956), 321–322.

For the polynomial $Q_n(x)$ of the preceding review put

$$Q_n=2^n\binom{2n}{n}^{-1}W_n,\quad W(t)=\sum_0^\infty W_n t^n/n!.$$

Then if $W=(1-t)^{-2x-1}w,\ z=t^2$, it follows that

$$z(z-1)\frac{d^2w}{dz^2}+[1-(1-2x)z]\frac{dw}{dz}-x^2w=0,$$

so that $w=F(-x, -x; 1; z)$. This leads to the explicit formula

$$Q_n(x)=2^n n!\binom{2n}{n}^{-1}\sum_{2r\leq n}\binom{2x+n-2r}{n-2r}\binom{x}{r}^2.$$

*L. Carlitz* (Durham, N.C.).

**Sierpiński, W. Sur une propriété de la fonction $\varphi(n)$.** Publ. Math. Debrecen 4 (1956), 184–185.

For every integer $k$, the equation $\varphi(x+k)=\varphi(x)$ has a solution. Given $m$, there exist integers $k\geq 1$ such that $\varphi(x+k)=\varphi(x)$ has more than $m$ solutions.

*H. S. Zuckerman* (Seattle, Wash.).

**Schinzel, A.; and Wang, Y. A note on some properties of the functions $\varphi(n)$, $\sigma(n)$ and $\theta(n)$.** Bull. Acad. Polon. Sci. Cl. III. 4 (1956), 207–209.

Results are announced without proof of extensions and refinements of previous results of A. Schinzel [same Bull. 2 (1954), 467–469; 3 (1955), 415–419; MR 16, 675; 17, 461]. *I. Niven* (Eugene, Ore.).

**Kubilyus, I. P. Probabilistic methods in the theory of numbers.** Uspehi Mat. Nauk (N.S.) 11 (1956), no. 2(68), 31–66. (Russian)

This is an enlarged version of a report delivered before a Conference on the Theory of Probability and its Applications, held in Leningrad in 1955. It is divided into three main sections; the first is a full account of strongly additive arithmetic functions, the second of Linnik's "Large Sieve" and its generalisations, and the third of the Arithmetic of Quaternions. Appended is an exhaustive bibliography. [For an earlier and less comprehensive account of this subject see Kac, Bull. Amer. Math. Soc. 55 (1949), 641–665; MR 11, 161.]

*H. Halberstam* (Exeter).

**See also:** Surányi, p. 4; Newman, p. 11; Rédei, p. 15; Lehmer, p. 74.

## *Analytic Theory of Numbers*

**Buhštab, A. A. On an additive representation of integers.** Moskov. Gos. Ped. Inst. Uč. Zap. 71 (1953), 45–62. (Russian)

Let $T_{\alpha,\beta}(N)$ denote the number of representations of the positive integer $N$ in the form $n'+n''$, where all prime factors of $n'$ are greater than $N^{1/\alpha}$ and all prime factors of $n''$ are greater than $N^{1/\beta}$. In an earlier paper [C. R. (Dokl.) Acad. Sci. URSS (N.S.) 29 (1940), 544–548; MR 2, 348] the author has used an improved form of Brun's method to prove that if $N$ is large

$$(*)\qquad T_{5,5}(N)\geq 0.4N(\ln N)^{-2}$$

(and thus that $N$ is expressible as a sum of two positive integers each containing at most four prime factors). Here he uses results from another paper [Mat. Sb. N.S. 28(70) (1951), 165–184; MR 13, 626] to prove that whenever $\beta\geq\alpha$ we have

$$T_{\alpha,\beta}(N)=CN(\ln N)^{-2}\psi(\alpha, \beta; N)\prod_{p|N}\{1+1/(p-2)\}+O(N(\ln N)^{-3}),$$

where $C$ is a positive absolute constant, $p$ runs over the odd primes, and

$$|\psi(\alpha, \beta; N)/(\alpha\beta)-1|<\exp\{-\alpha(\ln\alpha+\ln\ln\alpha-1-\ln 2+\gamma\ln\ln\alpha/\ln\alpha)\},$$

$\gamma$ being an absolute constant which is unfortunately not calculated. (Even if $\gamma$ were calculated explicitly, the value obtained would probably not be large enough to establish a result analogous to (*) for $T_{4,4}(N)$.) *P. T. Bateman.*

Have

★ **Erdös, P. Problems and results in additive number theory.** Colloque sur la Théorie des Nombres, Bruxelles, 1955, pp. 127–137. Georges Thone, Liège; Masson and Cie, Paris, 1956.

"In this lecture I will discuss several problems in additive number theory. They will not have much in common, except that they are all combinatorial in nature and that probability theory can be applied with advantage to most of them." Much of the paper is devoted to discussing known results and conjectures, but proofs of several new results are outlined. For example, let $a_1 < a_2 < \cdots$ be an increasing sequence of positive integers and define $f(n)$ by the power-series identity

$$\sum_{n=0}^{\infty} f(n)x^n = \Big(\sum_{k=1}^{\infty} x^{a_k}\Big)^2.$$

In an earlier paper [Acta Sci. Math. Szeged **15** (1954), 255–259; MR **16**, 336] Erdös proved the existence of a sequence for which (*) $0 < f(n) < c \log n$ for large $n$. Here he gives another proof by constructing a suitable probability measure on the space of all increasing sequences of positive integers and then using probability theory to show that (*) is true for almost all sequences relative to this measure. *P. T. Bateman* (Princeton, N.J.).

**Roux, Delfina. Sulla distribuzione degli interi rappresentabili come somma di due quadrati.** Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **21**(**90**) (1956), 137–140.

Let $A(\xi)$ denote the number of integers representable as the sum of two squares and not exceeding $\xi$. Making a simple application of the sieve method of Viggo Brun the author derives the upper bound: $A(\xi) < k\xi/(\log \xi)^{\frac{1}{2}}$ for $\xi > \xi_1$. *A. L. Whiteman* (Los Angeles, Calif.).

**Hornfeck, Bernhard. Dichtentheoretische Sätze der Primzahltheorie.** Monatsh. Math. **60** (1956), 96–109.

Let $\mathfrak{M}$ be a set of positive integers. Then $\mathfrak{M}^k$ is defined as the set of all integers which can be represented in the form $m_1 \cdots m_k$, $m_i \in \mathfrak{M}$ $(i=1, \cdots, k)$. Denote by $M(x)$ the number of $m \in \mathfrak{M}$, $m \leq x$. If $\phi(x) \to \infty$ as $x \to \infty$ and the limit $\lim_{x=\infty} M(x)/\phi(x) = \tau$ exists and is positive, then $\tau$ is said to be the $\phi(x)$-density of $\mathfrak{M}$.

The present paper is concerned with relations between the densities of a set and its powers. As a typical result we may quote the following. Let $\mathfrak{T}$ be a set of primes and suppose that it has $(x/\log x)$-density $\tau$; then $\mathfrak{T}^k$ has $\{x(\log\log x)^{k-1}/(k-1)! \log x\}$-density $\tau^k$. When $\mathfrak{T}$ is the set of all primes, this result reduces to a well-known asymptotic formula of Landau [Bull. Soc. Math. France **28** (1900), 25–38].

In addition to theorems about subsets of primes, the author also obtains more general relations involving sets of pairwise coprime integers. *L. Mirsky* (Sheffield).

**Yamamoto, Koichi. Theory of arithmetic linear transformations and its application to an elementary proof of Dirichlet's theorem.** J. Math. Soc. Japan **7** (1955), 424–434.

The author gives a proof of Dirichlet's theorem on the infinitude of primes in arithmetic progression which is essentially the same as given previously by the reviewer [Ann. of Math. (2) **52** (1950), 231–243, pp. 232–236; MR **12**, 81]. Also included is a simplifying remark concerning the original method of A. Selberg [ibid. **50** (1949), 297–304; MR **10**, 595]. *H. N. Shapiro* (New York, N.Y.).

**Erdös, P. On pseudoprimes and Carmichael numbers.** Publ. Math. Debrecen **4** (1956), 201–206.

A composite number $n$ is called a pseudoprime if $2^n \equiv 2 \pmod{n}$; it is called a Carmichael number if $a^n \equiv a \pmod{n}$ for all numbers $a$ relatively prime to $n$. $P(x)$ and $C(x)$ denote the number of pseudoprimes $\leq x$ and the number of Carmichael numbers $\leq x$, respectively. Improving a result of Knödel [Archiv. Math. **4** (1953), 282–284; MR **15**, 289] the author shows that

$$P(x) < x \exp\left(-c_1(\log x \log\log x)^{\frac{1}{2}}\right),$$

$$C(x) < x \exp\left(-c_2 \log x \log\log\log x/\log\log x\right),$$

with positive constants $c_1$, $c_2$. He conjectures that $(\log C(x))/\log x \to 1$ as $x \to \infty$. *N. G. de Bruijn.*

**Obláth, Richard. Sur la répartition des nombres sans diviseur quadratique.** Publ. Math. Debrecen **4** (1956), 131–134.

The author proves that if $x$ is sufficiently large and if $n = O(x)$ then the number of square-free integers within each of the intervals $[1, x]$, $[x, 2x]$, $\cdots$, $[(n-1)x, nx]$, is asymptotically uniform and equal to $6x/\pi^2$. Assuming the Riemann hypothesis, the conclusion still holds if the condition $n = O(x)$ is replaced by $n = O(x^{\frac{1}{2}-\epsilon})$. In line 6 of p. 132, read $o(\sqrt{mx})$ instead of $o(\sqrt{mx})$.

*T. M. Apostol* (Pasadena, Calif.).

**Selberg, Sigmund. Über eine Vermutung von P. Turan.** Norske Vid. Selsk. Forh., Trondheim **29** (1956), 33–35.

Let

$$L(x) = \sum_{n=1}^{x} \frac{\lambda(n)}{n}, \quad H(x) = \sum_{n=1}^{x} L(n),$$

where $\lambda(n)$ is Liouville's function, i.e. $\lambda(n) = (-1)^{\Omega(n)}$, where $\Omega(n)$ is the number of (equal or distinct) prime factors of $n$. Let $H$ and $h$ be the upper and lower limits, respectively, of $H(x)\sqrt{x}$ as $x \to \infty$; and let $a = -2/\zeta(\frac{1}{2})$ $(=1.37$ approximately). The author notes first that, by a classical theorem of Landau, $H \geq a \geq h$. Turán's conjecture (T) is that $L(x) > 0$ $(x \geq 1)$ [Danske Vid. Selsk. Mat.-Fys. Medd. **24** (1948), no. 17; MR **10**, 286]. The aim of this paper is to show that $T$ implies that $h \geq \frac{4}{3}$ and $H < 10$. The method is elementary and is based on properties of the function

$$\sigma_x(n) = \sum_{t=1}^{x} \left\{ n\left[\frac{t}{n}\right] - (n+1)\left[\frac{t}{n+1}\right] \right\}$$

introduced by E. Jacobsthal.

{Notes by the reviewer: (1) By a classical method of E. Schmidt and Landau the inequalities $H \geq a \geq h$ may be strengthened to $H \geq a + c > a - c \geq h$, where $c$ is a (finite or infinite) positive constant depending on the complex zeros of $\zeta(s)$. (2) As the author observes, the hypothesis (S) that $H(x)/\sqrt{x}$ is bounded above or below implies (like T) the Riemann hypothesis, i.e. that the complex zeros of $\zeta(s)$ are of the form $\frac{1}{2} \pm i\gamma_n$ $(\gamma_n > 0)$ (and, it may be added, are all simple, so that the $\gamma_n$ are distinct). Now $T$ implies $S$ (obviously, with 0 as a lower bound), and indeed more (by the main result of this paper), so $S$ is to this extent weaker than $T$. It is also more plausible in that it does not (like $T$) compel us to accept the (improbable) conclusion that the $\gamma_n$ are linearly dependent [see the review quoted above].} *A. E. Ingham* (Cambridge, England).

**Bochner, S.; and Chandrasekharan, K. On Riemann's functional equation.** Ann. of Math. (2) **63** (1956), 336–360.

The authors consider solutions $\{\varphi(s), \psi(s)\}$ of the functional equation (1) $\pi^{-s/2}\Gamma(s/2)\varphi(s)=\pi^{-\frac{1}{2}(\delta-s)}\Gamma\frac{1}{2}(\delta-s)\psi(\delta-s)$ such that there are Dirichlet series for $\varphi$ and $\psi$:

$$\varphi(s)=\sum_1^\infty a_n\lambda_n^{-s},\ \psi(s)=\sum_1^\infty b_n\mu_n^{-s}\ (s=\sigma+i\tau)$$

converging absolutely for $\sigma>\alpha>0$, $\sigma>\beta>0$ respectively. Here $\delta>0$ as well as the increasing sequences $\lambda_n\to\infty$, $\mu_n\to\infty$ are supposed to be given. They further suppose the existence of a bounded closed set $S$ in the $s$-plane, symmetric relative to the line $\sigma=\delta/2$, $-\infty<\tau<+\infty$ and the existence of a holomorphic function $\chi(s)$ in the exterior $R$ of $S$ which in a right half-plane coincides with the left hand side of (1) and in a left half-plane with the right hand side of (1) and such that

$$\lim_{|\tau|\to\infty}\chi(\sigma+i\tau)=0,$$

uniformly in every bounded interval $\sigma_1\leqq\sigma\leqq\sigma_2$, $-\infty<\sigma_1<\sigma_2<+\infty$. They obtain an upper bound for the number of linearly independent solutions which depends on the distribution of the sequences $\{\lambda_n\}$ and $\{\mu_n\}$. After introducing certain further restrictions the only solutions that can occur are $\zeta(s)$, $(2^s-1)\zeta(s)$, $(2^{1-s}-1)\zeta(s)$, $2^{s-1}L(s-1)$. The method used is essentially that of Siegel for the proof of Hamburger's well-known theorem on the solution of Riemann's functional equation for $\zeta(s)$.

*H. D. Kloosterman* (Leiden).

**Chandrasekharan, Komaravolu; et Mandelbrojt, Szolem. Sur l'équation fonctionnelle de Riemann.** C. R. Acad. Sci. Paris 242 (1956), 2793–2796.

The authors announce a most interesting proposition in a topic recently initiated, more or less, by this reviewer [Ann. of Math. (2) **53** (1951), 332–363; MR 13, 920 and very recently continued in the paper reviewed above. Take two Dirichlet series

$$\varphi(s)=\sum a_n\lambda_n^{-s},\ \psi(s)=\sum b_n\mu_n^{-s}\ (s=\sigma+i\tau)$$

each absolutely convergent somewhere, $\lambda_n>0$, $\mu_n>0$, $\lambda_n\uparrow\infty$, $\mu_n\uparrow\infty$, and for some $\delta>0$ they shall satisfy a functional equation

$$(*)\qquad \pi^{-s/2}\Gamma(\tfrac{1}{2}s)\varphi(s)=\pi^{-\frac{1}{2}(\delta-s)}\Gamma(\tfrac{1}{2}(\delta-s))\psi(\delta-s)$$

in the sense that there exists, in a domain $|s|>R$, a holomorphic function $\chi(s)$ which in some right half-plane coincides with the left side of (*) and in some left half-plane with its right side, and for which $\lim_{|\tau|\to\infty}\chi(\sigma+i\tau)=0$ uniformly in every bounded interval $\sigma_1\leqq\sigma\leqq\sigma_2$.

The authors deal only with the parameters $\delta=1$ and $=3$ which "correspond" to the classical cases $\varphi(s)=\psi(s)=\zeta(s)$ and $L(s)$, in substance, and their statement is as follows. If only the exponents in one of the Dirichlet series are spaced apart, say $\lim\inf_{n\to\infty}(\mu_{n+1}-\mu_n)<\infty$, than those in the other are only one step from being equidistant: there exists an $l>0$ such that for any $c\geqq0$ if one denotes the exponents $\{\lambda_m\}$ in $c<\lambda\leqq c+l$ by $\lambda_q\ \lambda_{q+1},\cdots,\lambda_{q+r}$, then any other $\lambda_n$ is a linear combination $p_0^{(n)}\lambda_q+p_1^{(n)}\lambda_{q+1}+\cdots+p_k^{(n)}\lambda_{q+r}$ with integer coefficients $p_k^{(n)}$.

Furthermore, the function $\sum b_n e^{-2\pi\mu_n s}$, $\sigma>0$, of which it was known that it has a univalent continuation into $\sigma<0$ (except for poles at $\pm i\lambda_n$ and perhaps also at 0), is such that in $\sigma<0$ it has an expansion $\sum_0^\infty\gamma_n e^{2\pi\mu_{-n}s}$, $0<\mu_{-n}\uparrow\infty$, likewise. *S. Bochner* (Princeton, N.J.).

**Pyateckiĭ-Šapiro, I. I. On the theory of abelian modular functions.** Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 973–976. (Russian)

The author gives a unified approach to the modular functions of Hilbert, Siegel [Math. Ann. **116** (1939), 617–657; MR 1, 203] and Braun [Ann. of Math. (2) **50** (1949), 827–855; MR 11, 333]. The approach is based on a connection between E. Cartan's first three bounded symmetric domains [see Siegel, Analytic functions of several complex variables, Inst. Advanced Study, Princeton, N.J.,1950; MR 11, 651] and the irreducible components of the multiplier algebra $\mathfrak{A}$ [see Albert, Ann. of Math. **35** (1934), 500–515]. These algebras are rational square matrices of order $2p$ for which every element $A$ determines a complex square matrix $\alpha$ of order $p$ such that $\omega A=\alpha\omega$. Here $\omega$ is a complex matrix of order $p\times2p$ for which $\omega R_0\omega'=0$, $i\bar\omega R_0\omega'>0$, where, finally, $R_0$ is a rational skew-symmetric matrix of order $2p$ for which the transformation $R_0A'R_0^{-1}$ maps $\mathfrak{A}$ onto itself. The modular group consists of (say) integral square matrices $U$ of order $2p$ for which $U'R_0U=R_0$, $U'A=AU'$ for all $A$ in $\mathfrak{A}$. Modular forms of weight $m$ on $\omega$ are defined by the properties $f(\alpha\omega)=|\alpha|^{-m}f(\omega)$, $f(\omega U)=f(\omega)$, (with suitable boundedness) and the set of equivalence classes $\omega\sim\alpha\omega$ determines the domain with irreducible components identified as Cartan's third, first, or second domain according as the irreducible components of the real envelope of $\mathfrak{A}$ are reals, complex numbers, or quaternions respectively. General proofs on the existence of modular functions are also outlined. *Harvey Cohn* (St. Louis, Mo.).

**See also:** Richert, p. 31.

## *Theory of Algebraic Numbers*

Have

★ **Pisot, Ch. Sur une famille remarquable d'entiers algébriques formant un ensemble fermé.** Colloque sur la Théorie des Nombres, Bruxelles, 1955, pp. 77–83. Georges Thone, Liège; Masson and Cie, Paris, 1956.

The author recounts, with ample bibliography, the story of the closed set $S$ of algebraic numbers $\theta$ $(>1)$ with conjugates all $>1$ in modulus. Among the most recent work is the fuller realization of a relation with sets of uniqueness of Fourier series, [Salem and Zygmund, C. R. Acad. Sci. Paris **240** (1955), 2281–2283; MR **17**, 150], the author's exploration of small values in $S'$ (the derived set) [Dufresnoy and Pisot, Bull. Sci. Math. (2) **79** (1955), 54–64; MR **17**, 463], and a generalization of the set $S$ by Doubrère [C. R. Acad. Sci. Paris **240** (1955), 2111–2113; MR 16, 908].

*Harvey Cohn* (St. Louis, Mo.).

**Duyčev, J. On prime ideals of degree 1.** Acta Math. Acad. Sci. Hungar. 7 (1956), 71–73. (Russian. English summary)

The author, by choice of $q$, shows the infiniteness of the set of prime ideals $\mathfrak{p}$ dividing the aggregate $\eta=M\xi+q$, where $\xi$ is a given primitive number of a finite extension of the rationals, where $M=N(\xi)D(\xi)$ ($N$ and $D$ denoting norm and discriminant), and where $(q, M)=1$. Thus $\mathfrak{p}|\eta$, $\mathfrak{p}\nmid D(\eta)$, and $\eta$ is primitive. As a consequence the author easily concludes $\mathfrak{p}$ is of degree one, proving, without benefit of zeta-function, that there are infinitely many such prime ideals. *Harvey Cohn.*

Have

★ **Bruck, R. H. Computational aspects of certain combinatorial problems.** Proceedings of Symposia in Applied Mathematics. Vol. VI. Numerical analysis, pp. 31–43. Published by McGraw-Hill Book Company, Inc., New York, 1956 for the American Mathematical Society, Providence, R. I. $9.75.

This paper is a discussion of a famous problem in cyclotomy and the role that experimental computing can play in the consideration of unsolved aspects of the problem. More explicitly, the problem is to get precise formulas for the Gaussian integers $(u, v)_m$ defined in terms of a prime $p=km+1$ and a primitive root $g$ of $p$ as the number of solutions $(x, y)$ of the trinomial congruence

$$g^{u+mx}+g^{v+my}+1\equiv 0 \pmod{p}.$$

The kind of formula desired is indicated by the following given by Gauss but misprinted in the paper.

$$18(1, 1)_3=2p-4-a+9b,\ 18(2, 2)_3=2p-4-a-9b,$$

where $4p=a^2+27b^2$, $a\equiv 1 \pmod 3$, conditions which determine everything uniquely except the sign of $b$. This ambiguity reflects the dependence of the symbol $(u, v)_3$ on the choice of primitive root $g$. After some remarks in general about the direct computation of the symbol $(u, v)_m$ by punched card methods and the connection of the problem with the so-called Jacobi sums the author proceeds to discuss the state of the problem for the specific values of $m=3, 4, 5, 6, 8, 9, 10, 12, 16, 20$. The problem may be regarded as solved for $m\leq 8$. In general we are looking for a representation of $(u, v)_m$ as a linear combination of $p$ and of parts in the various quadratic partitions of $p$ in one or more of the forms.

$$p=a^2+4b^2,\ a\equiv 1 \pmod 4),\ 4p=a^2+27b^2,\ a\equiv 1 \pmod 3),$$
$$p=a^2+8b^2,\ a\equiv 1 \pmod 4),\ p=a^2+3b^2,\ a\equiv 1 \pmod 3),$$
$$16p=a^2+125b^2+50c^2+50d^2,$$
$$ab=c^2+4cd-d^2,\ a\equiv 1 \pmod 5),$$

which are regarded as known. The dependence upon $p$ appears at $m=6$. That is, there may be two or more sets of formulas each to be used only for $p$'s belonging to one of a given set of airthmetic progressions. That such a representation exists in general is shown to be false for the case $m=20$. Indeed it is false for $m=16$. What further basic elements are needed in these cases is unknown.

*D. H. Lehmer* (Berkeley, Calif.).

**Carlitz, L. The number of solutions of a particular equation in a finite field.** Publ. Math. Debrecen **4** (1956), 379–383.

Let $q=p^n$ and $m|q+1$. The author determines the number of solutions of the equation

$$\alpha_1\xi_1^m+\cdots+\alpha_s\xi_s^m=\alpha,$$

where $\alpha_i, \xi_i, \alpha \in GF(q^2)$ and $\alpha_i\neq 0$. In the special case $\alpha_1=\cdots=\alpha_s=\alpha=1$, his result reduces to a theorem of Faircloth [Canad. J. Math. 4 (1952), 343–351; MR 13, 915]. The proof makes use of a formula due to Stickelberger [Math. Ann. 37 (1890), 321–367].

*A. L. Whiteman* (Los Angeles, Calif.).

**Hasse, Helmut. Die dyadische Einseinheitenoperatorgruppe zum Körper der $2^n$-ten Einheitswurzeln nebst Anwendung auf die Klassenzahl seines grössten reellen Teilkörpers.** Rev. Fac. Sci. Univ. Istanbul. Sér. A. **20** (1955), 7–16. (Turkish summary)

This interesting note contains a new proof for the fact that the class number of the field of all $2^n$th roots of unity $P_n=Q(\zeta)$ is odd. As in the older proofs which use signatures of units [see H. Hasse, Über die Klassenzahl abelscher Zahlkörper, Akademie-Verlag, Berlin, 1952; MR 14, 141] reduction is made to the statement that the class number $\tilde{h}$ of the maximal real subfield $\tilde{P}_n$ of $P_n$ is odd. Let $\lambda$ denote the prime divisor of 2 in $\tilde{P}_n$. The author considers the units modulo a sufficiently high power of $\lambda$ and shows that the maximal count of independent units modulo this power for the full unit group $\tilde{E}$ is already reached for the group of circular units $\tilde{E}_0$. It turns out to be convenient to carry this investigation directly to the corresponding unit groups $E$, $E_0$ in $P_n$. For this purpose the structure of the group of all 1-units $H$ and that of $H/H^2$ in the $(1-\zeta)$-adic completion $\bar{P}_n$ of $P_n$ is determined as an operator group with respect to the dyadic group ring of the Galois group $\{s\}$ of $P_n/P_2$, and a suitable system of 4 generators with their relations is set up by means of congruence properties. These congruence yield that the generators $\zeta$, $[(1-\zeta^5)/(1-\zeta)]^{(s-1)^\nu}$ $(0\leqq\nu\leqq 2^{n-2}-2)$ of $E_0$ remain independent in $H/H^2$ whence $[E_0:E_0^2]=2^{n-2}=[E:E^2]$ so that $[E:E_0]$ must be odd, and as a simple consequence that $\tilde{h}=[\tilde{E}/\{-1\}:\tilde{E}_0/\{-1\}]$ is odd.

*O. F. G. Schilling* (Chicago, Ill.).

**Inkeri, K. Über die Klassenzahl des Kreiskörpers der $l$ten Einheitswurzeln.** Ann. Acad. Sci. Fenn. Ser. A. I. no. **199** (1955), 12 pp.

Suppose that $l$ is a prime larger than 3. Pick for $r$ a primitive root modulo $l$ and denote by $r_i$ the smallest positive remainder of $r^i$ modulo $l$. Furthermore let $m=(l-1)/2$ and denote by $Z$ a primitive $(l-1)$-st root of unity. Moreover set $f(x)=\sum_{i=0}^{l-2} r_i x^i$. The author improves a result of Vandiver for the first factor

$$h_1=(-1)^m(1/2l)^{m-1}\prod_s f(Z^s)$$

of the class number of the field of all $l$th roots of unity, $s$ running over all odd numbers between 1 and $l-2$. He fin s $h_1\equiv K_n l\prod_s B(s) \pmod{l^n}$ where $K_n$ is a rather complicated integer which, however, can be easily shown to be relatively prime to $l$, $B(s)$ is the Bernoulli number of index $(sl^{n-1}+1)/2$. The integral character of $h_1$ is established by means of a determinant involving the $r_i$ and the integers $(rr_i-r_{i+1})/1$ for suitable $r$. Finally the author presents a simplified form of another result of Vandiver concerning the divisibility by $l$ of the second factor of the class number.

*O. F. G. Schilling* (Chicago, Ill.).

**Dénes, Peter. Über den zweiten Faktor der Klassenzahl und den Irregularitätsgrad der irregulären Kreiskörper.** Publ. Math. Debrecen **4** (1956), 163–170.

Let $p$ denote an irregular prime, $\zeta$ a primitive $p$th root of unity, $R(\zeta)$ the cyclomotic field generated by $\zeta$, $\lambda=1-\zeta$, $\mathfrak{l}=(\lambda)$, $h_2$ the second factor of the class number of $R(\zeta)$. Put $h_2=p^{f_0}q$, $p\nmid q$. Vandiver [Proc. Nat. Acad. Sci. U.S.A. 16 (1930), 743–749] proved that a necessary and sufficient condition that $p|h_2$ is that one of the units

$$E_i=\varepsilon^{G_i}\ (i=1, \cdots, \tfrac{1}{2}(p-3))$$

be the $p$th power of a unit of $R(\zeta)$. Here

$$\varepsilon=\sqrt{\frac{(1-\zeta^r)(1-\zeta^{-r})}{(1-\zeta)(1-\zeta^{-r})}},$$

$r$ is a primitive root (mod $p$), $s$ denotes the substitution $\zeta\to\zeta^r$ and

$$G_i=r^{p^s}+sr^{p^s-2i}+s^2r^{p^s-4i}+\cdots+s^{\frac{1}{2}(p-3)}r^{p^s-(p-3)i}.$$

The object of the present paper is to determine the value of $f_2$. The following results are proved.

I. Let the primitive root $r$ satisfy

$$r^{p-1}\equiv 1 \pmod{p^F} \quad (F>f_2+1)$$

and let

$$E_i=\varepsilon^{S_i},\ S_i=1+sr^{-2i}+s^2r^{-4i}+\cdots+s^{\frac{1}{2}(p-3)}r^{-(p-3)i};$$

also assume that

$$E_i=\delta_i p^{v_i} \quad (\delta_i \in R(\zeta);\ i=1, \cdots, \tfrac{1}{2}(p-3)).$$

Then

$$(*) \qquad f_2=\sum_{1}^{\frac{1}{2}(p-3)} v_i.$$

II. Define $u_i$ by means of

$$B_{ip^j}\equiv 0 \pmod{p^{2j+1}} \quad (j=0, \cdots, u_i-1);$$
$$B_{ip^{u_i}}\not\equiv 0 \pmod{p^{2u_i+1}}$$

define $u_i'$ by means of

$$\varrho_i\equiv m_i+n_i\lambda^{u_i'(p-1)+2i} \pmod{\mathfrak{l}^{u_i'(p-1)+2i+1}},$$

where the $\varrho_i$ are a fundamental set of units, $m_i$ and $n_i$ are rational integers prime to $p$. The numbers $u_i$ and $u_i'$, which were previously introduced by the author, are called the $p$-character of the Bernoulli numbers and of the fundamental units, respectively. The explicit result for $f_2$ reads

$$(**) \qquad f_2=\sum_{L}^{\frac{1}{2}(p-3)} (u_i-u_i').$$

Indeed it is shown that $v_i=u_i-u_i'$ so that (**) follows from (*).

In his earlier paper [Publ. Math. Debrecen **3** (1953), 17–23; MR **15**, 686] the author defined $w$ as the greatest of the numbers $u_i$; he called $w$ the Irregularitätsgrad of the prime $p$. He raised the question of the finiteness of $w$ for given $p$ and promised to prove this assertion in a later paper. This promise is repeated in the present paper. However, it is pointed out that the finiteness of $w$ is a consequence of Furtwängler's conjecture concerning the class field tower: If $k$ is an algebraic number field, $K_1$ the class field of $k$, $K_2$ the class field of $K_1$, and so on, then according to the conjecture, there is an integer $n$ such that $K_{n+1}$ is identical with $K_n$. *L. Carlitz* (Durham, N.C.).

See also: Reiner, p. 7; Moriya, p. 8; Salem, p. 23; Taussky, p. 71.

## *Geometry of Numbers*

**Schmidt, W. Eine neue Abschätzung der kritischen Determinante von Sternkörpern.** Monatsh. Math. **60** (1956), 1–10.

Let $S$ be a finite star body in $n$-dimensional space which is symmetric with respect to the origin. Let $\Delta(S)$ denote the critical determinant of $S$ and $V(S)$ its volume. In the above paper a system of constants $A(n, K)$ is constructed dependent on the first $K$ prime numbers for which $V(S)/\Delta(S)\geqq 2\zeta(n)A(n, K)$. For $n>2$ $A(K, n)$ exists which is greater than 1. This shows that the bound $2\zeta(n)$ given by the Minkowski-Hlawka theorem [Hlawka, Math. Z. **49** (1943), 285–312; MR **5**, 201] is not the best possible. An essential link in the derivation of the result is a limit obtained by Rogers by analysis [Ann. of Math. (2) **48** (1947), 994–1002; MR **9**, 270]. Apart from this the proofs depend only on elementary congruence computations. *D. Derry* (Vancouver, B.C.).

**Rogers, C. A. The number of lattice points in a set.** Proc. London Math. Soc. (3) **6** (1956), 305–320.

Let $\varrho(x_1, x_2, \cdots, x_n)=\varrho(x)$ be a non-negative measurable function and $\Lambda$ a lattice defined in $n$-space. A function $\varrho^*(x)$, called the spherical symmetrization of $\varrho(x)$, is defined as the greatest lower bound of the numbers $\varrho$ with the property that the measure of the set of points $x'$ with $\varrho(x')>\varrho$ does not exceed the measure of the set of points $y$ with $|y|\leqq|x|$. Let $\mu(\Lambda)$ be the measure defined for the space of lattices with determinant by Siegel [Ann. of Math. (2) **46** (1945), 340–347; MR **6**, 257] and $\varrho(\Lambda)=\sum_{0\neq x\in\Lambda}\varrho(x)$. The author shows that

$$\int\{\varrho(\Lambda)\}^k d\mu(\Lambda)\leqq\int\{\varrho^*(\Lambda)\}^k d\mu(\Lambda), \text{ for } k=1, 2, 3,$$

and provided that the set of points $x$ with $\varrho(x)>c$ is convex for each constant $c$, for $k=1, 2, 3, \cdots$.

If $\varrho(x)$ is specialized to be the characteristic function of a Borel set with measure $V$, symmetric with respect to the origin, then it is shown that for $n\geqq n(k)$,

$$2^k e^{-\frac{1}{2}V}\sum_{r=0}^{\infty}\frac{r^k}{r!}(\tfrac{1}{2}V)^r\leqq\int\{\varrho(\Lambda)\}^k d\mu(\Lambda)\leqq$$
$$2^k e^{-\frac{1}{2}V}\sum_{r=0}^{\infty}\frac{r^k}{r!}(\tfrac{1}{2}V)^r+f(n, k, V),$$

where $\lim_{n\to\infty} f(n, k, V)=0$. This result is applied to obtain the improvement of the Minkowski-Hlawka theorem that

$$\frac{V(S)}{\Delta(S)}\geqq\frac{\sqrt{n}}{3},$$

provided $n$ is sufficiently large. This result, for large $n$, is stronger than another improvement of the same theorem recently given in the paper reviewed above. *D. Derry* (Vancouver, B.C.).

**Hejtmanek, Johann. Über eine Klasseneinteilung der Sternkörper.** Monatsh. Math. **60** (1956), 11–20.

Let $k$ be a given non-negative integer. The author constructs a star body in Euclidean $n$-space, symmetric with respect to the origin, which admits a critical lattice with exactly $k$ point pairs on the boundary of the star body, provided (1) $n=2$, $k\geqq 2$, in which case the constructed star body is bounded, or (2) a star body $K_n$ exists which is both completely reducible and fully automorphic. Star bodies $K_n$, $2\leqq n\leqq 4$, which satisfy the necessary conditions have been given by Davenport and Rogers [Philos. Trans. Roy. Soc. London. Ser. A. **242** (1950), 311–344; MR **12**, 394]. Thus the problem is solved for these values of $n$.

It is shown that the Minkowski-Hlawka theorem implies that a star body of finite volume must be of the finite type, i.e. it must admit at least one lattice. *D. Derry.*

**Mahler, K. A property of the star domain $|xy|\leqq 1$.** Mathematika **3** (1956), 80.

The author shows that each star polygon with finitely many sides which is contained in the star domain $K$: $|xy|\leqq 1$ is of smaller lattice determinant than $K$. *C. G. Lekkerkerker* (Amsterdam).

**Bambah, R. P. Divided cells.** Res. Bull. Panjab Univ. no. **81** (1955), 173–174.

A proof on B. N. Delone's theorem that an inhomogene-

ous plane lattice contains a divided cell [Izv. Akad. Nauk SSSR. Ser. Mat. **11** (1947), 505–538; MR **9**, 334; see also E. S. Barnes and H. P. F. Swinnerton-Dyer, Acta Math. **92** (1954), 199–234; MR **16**, 802]. *J. W. S. Cassels.*

**Birch, B. J.; and Swinnerton-Dyer, H. P. F. On the inhomogeneous minimum of the product of $n$ linear forms.** Mathematika **3** (1956), 25–39.

The authors develop new techniques to attack Minkowski's conjecture that the inhomogeneous minimum of the product of $n$ linear forms $L_j$ in $n$ variables $x_j$ of determinant $\Delta\neq 0$ is at most $2^{-n}|\Delta|$, with equality only when the set of forms is equivalent to $L_j=\lambda_j x_j$ for constants $\lambda_j$ under an integral unimodular transformation. They first show that the conjecture is certainly true for all sets of forms sufficiently near to $L_j=\lambda_j x_j$ in the sense of Mahler's topology for lattices. Secondly, they show that the conjecture is generally true for $n$ forms if it is true for all smaller numbers of forms and also for the very special sets of $n$ forms whose product has an attained strictly positive homogeneous minimum. Finally they give a verification of the conjecture for $n=3$ independent of the theory of quadratic forms used by earlier workers [R. Remak, Math. Z. **17** (1923), 1–34; **18** (1923), 173–200; H. Davenport, J. London Math. Soc. **14** (1939), 47–51].
*J. W. S. Cassels* (Cambridge, England).

**Postnikov, A. G. Properties of solutions of Diophantine inequalities in the field of formal power series.** Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 21–22. (Russian)

The author states without proof some results analogous to those of the geometry of numbers about the ring of formal power series over a field. Write $\omega(x)=\sum_{-l}^{\infty} a_u x^u$, where possibly $l<0$, and put $\|\omega(x)\|=l$ if $a_{-l}\neq 0$. For any $\omega_1(x)$, $\omega_2(x)$ and any integers $m$, $n$ there are polynomials $P(1/x)$, $Q(1/x)$, $R(1/x)$ in $1/x$ such that

$$\|P(1/x)\|\leqq m,\ \|Q(1/x)\|\leqq n,$$

$$\|P(1/x)\omega_1(x)+Q(1/x)\omega_2(x)-R(1/x)\|\leqq -m-n-1.$$

It is possible to choose $P$, $Q$, $R$ so that $\|P(1/x)\|=m$. Under certain restrictive conditions on the coefficients of $\omega_1(x)$, $\omega_2(x)$ there exist recurrence relations between the $P$, $Q$, $R$ obtained for differing values of $m$ and $n$.
*J. W. S. Cassels* (Cambridge, England).

**Szüsz, P. Beweis eines zahlengeometrischen Satzes von G. Szekeres.** Acta Math. Acad. Sci. Hungar. **7** (1956), 75–79. (Russian summary)

A simple proof based on continued fractions is given of the following theorem of G. Szekeres [J. London Math. Soc. **12** (1937), 88–93]: For any two real numbers $\alpha$, $\beta$ ($\alpha\neq\beta$) there exists a central-symmetric parallelogram of area $>K=2(1+5^{-\frac{1}{2}})$, having sides with slopes $\alpha$, $\beta$, and containing no lattice point besides the origin in its interior. *L. Tornheim* (Berkeley, Calif.).

★ **Roth, K. F. Rational approximations to algebraic numbers.** Colloque sur la Théorie des Nombres, Bruxelles, 1955, pp. 119–126. Georges Thone, Liège; Masson and Cie, Paris, 1956.

The author gives a simplified an readable outline of his proof of Siegel's conjecture on the approximation of algebraic numbers by rationals. The necessary bibliography can be found in his earlier work [Mathematika **2** (1955), 1–20, 168; MR **17**, 242]. *Harvey Cohn.*

# ANALYSIS

On order

## *Functions of Real Variables*

★ **Alexandroff, P. S. Einführung in die Mengenlehre und die Theorie der reellen Funktionen.** VEB deutscher Verlag der Wissenschaften, Berlin, 1956. xii+279 pp. DM 18.00.

Translation of: Vvedenie v obščuyu teoriyu množestv i funkciĭ, MR **12**, 682 which is Part One of Vvedenie v teoriu množestv i teoryu funkciĭ by P. S. Aleksandrov and A. N. Kolmogorov. [Gosudarstv. Izdat. Zehn.-Teor. Lit., Moscow-Leningrad, 1948].

**Shah, S. M. On a function of Ramanujan.** Amer. Math. Monthly **63** (1956), 407–408.

It has been conjectured by Ramanujan that the function

$$\phi(x)=\sum_{n=1}^{\infty}\frac{n^{n-2}}{(n-1)!}x^{n-1}e^{-nx}$$

is completely monotonic, in the sense that the first derivative is non-positive, the second derivative is non-negative, etc., for all $x\geqq 1$. This has been verified for the first four derivatives by Shah and Sharma [J. Univ. Bombay (N.S.) **17** (1948), part 3, sect. A, 1–4; MR **10**, 373]. In the present paper, the author considers some problems related to the above conjecture. He first proves two negative results, namely, that neither $x\phi(x)$ nor $e^x\phi(x)$ is completely monotonic for $x>1$. However, he then shows that both functions are absolutely monotonic (i.e., all derivatives are non-negative) in $y=xe^{-x}$ for all $x$ such that $0\leqq y<e^{-1}$ (all non-negative values of $x$ except $x=1$), and establishes the representation

$$e^x\phi(x)=\sum_{n=1}^{\infty}\frac{n^{n-2}}{(n-1)!}y^{n-1}$$

valid in the above range. *B. Lepson.*

**Johnson, R. A. A note concerning the Dirac delta function.** Proc. I.R.E. **44** (1956), 1058–1059.

**Szeptycki, P. On the identity of Morrey-Calkin and Schauder-Sobolev spaces.** Studia Math. **15** (1956), 123–128.

Many generalizations of the notion of differentiable function of several real variables have been made in connection with partial differential equations and the calculus of variations. Of this sort are the functions of class $\mathfrak{P}_\alpha$ of Calkin [Duke Math. J. **6** (1940), 170–186; MR **1**, 208] and Morrey [ibid. **6** (1940), 187–215; MR **1**, 209] and the $W_\alpha{}^1$ spaces of Schauder and Sobolev [Sobolev, Some applications of functional analysis in mathematical physics, Izdat. Leningrad. Gos. Univ., 1950; MR **14**, 565]. It is shown that the spaces $\mathfrak{P}_\alpha$ and $W_\alpha{}^1$ are identical for every $\alpha>1$. *W. H. Fleming* (Lafayette, Ind.).

**Jakubík, Ján. On uniform convergence of continuous functions.** Mat.-Fyz. Časopis. Slovensk. Akad. Vied **4** (1954), 154–161. (Slovak. Russian summary)

Let $M$ be the space of all continuous functions $x(t)$, $t\in\langle 0,1\rangle$. The author constructs proper subsets $M_1\subset M$

having the following three properties. (a) Every function of $M$ is pointwise limit of a sequence of functions of $M_1$. (b) $M_1$ is closed in the topology based on uniform convergence. (c) $M_1$ is a group with respect to the ordinary addition of functions as composition rule. The problem of finding such subsets $M_1 \subset M$ was posed by L. Mišik.

*C. Loewner* (Stanford, Calif.).

**Jarník, Vojtěch. Sur les fonctions linéairement dependantes.** Časopis Pěst. Mat. **80** (1955), 32–43. (Czech. Russian and French summaries)

Let $f_0(x), f_1(x), \cdots, f_{n-1}(x)$ be $n$ complex valued functions defined in the real interval $(a, b)$ and $n$-times differentiable at each point of the interval. Let us further assume that the rank $\nu$ of the Wronskian matrix of the $n$ functions is independent of $x$. The author proves then that the number of linearly independent relations of the form $c_0f_0+c_1f_1+\cdots+c_{n-1}f_{n-1}=0$ with constant coefficients $c_\mu$ holding in the whole interval $(a, b)$ is $n-\nu$. This generalizes a well-known criterion for linear dependence in so far as the usual assumption that the first $\nu$ rows of the Wronskian are linearly independent for each $x$ is dropped. *C. Loewner* (Stanford, Calif.).

**Salem, R. On monotonic functions whose spectrum is a Cantor set with constant ratio of dissection.** Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 49–55.

Let $P(\theta)$ be a perfect set of constant ratio of dissection $(\xi, 1-2\xi, \xi)$, $0<\xi<\frac{1}{2}$, $\theta=\xi^{-1}$. Associated with $P$ there is a "Lebesgue function" $f$ which is continuous non-decreasing and constant in every interval of the complement of $P$. Let $S$ be the set of real algebraic integers larger than 1 whose conjugates have moduli strictly less than 1. The author has shown that the Fourier Stieltjes coefficients of $f$ tend to zero if and only if $\theta \notin S$. In the present paper he considers the behaviour of the coefficients of more general functions having spectrum $P(\theta)$, $\theta \in S$, and proves the following results.

Let $G(x)$ be a continuous non-decreasing function $G(0)=0$, $G(1)=1$, and $F(x)=G(f)$. Then the Fourier-Stieltjes coefficients of $F$ do not tend to zero if i) the coefficients of $G$ tend to zero, ii) $G$ has a non-vanishing absolutely continuous part, iii) there is a non-constant absolutely continuous function $\psi$ such that the coefficients of $\psi(G)$ tend to zero.

The paper also includes an example of a continuous monotonic function whose Fourier-Stieltjes coefficients do not tend to zero and which has the same spectrum as a Lebesgue function with coefficients tending to zero.

*A. P. Calderón* (Cambridge, Mass.).

**Yang, Tsung-Pan B. Einige Bemerkungen über die Bairesche Eigenschaft.** Acta Math. Sinica **6** (1956), 83–91. (Chinese. German summary)

Durch die mit massgleicher Hülle u. massgleichem Kern in der Masstheorie analoge Begriffe in der Theorie der Baireschen Eigenschaft im metrischen Raum—Bairesche Hülle u. Bairescher Kern—, gewinnen wir einige Sätze der Baireschen Eigenschaft analog mit den von Rademacher [Monatsh. Math. Phys. **27** (1916), 183–291], Steinhaus [Fund. Math. **1** (1920), 93–104], und Wilkosz [ibid. **1** (1920), 82–92]. Zufällig lösen wir eine von Sierpiński aufgestellte Frage [Fund. Math. **20** (1933), S. 286] ohne Hypothese des Kontinuums. *Author's summary.*

**Szekeres, G. On a property of monotone and convex functions.** Proc. Amer. Math. Soc. **7** (1956), 351–353.

Let $f$ be a function with continuous second derivatives in an open interval $(a, b)$, $-\infty\leq a<b\leq\infty$. It is proved that if $f$ is strictly monotone increasing in $(a, b)$, then it has a representation in $(a, b)$ of the form $f(x)=\psi_1(\varphi(x))$, with $\varphi$ monotone increasing and convex, and with $\psi_1$ monotone increasing and concave. A necessary and sufficient condition is given in order that an $f$ that is also bounded can be represented in this form with bounded $\varphi$.

*F. F. Bonsall* (Newcastle-upon-Tyne).

**Sibagaki, Wasao. Uniform convergence and equicontinuity (fundamental theorems in elementary analysis).** Mem. Fac. Sci. Kyusyu Univ. Ser. A. **9** (1956), 111–133.

Esposizione sistematica di alcune proposizioni fondamentali dell'analisi matematica. *G. Scorza-Dragoni.*

**Marcus, S. Sur un problème de F. Hausdorff concernant les fonctions symétriques continues.** Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 201–205.

This note settles in the negative one of two problems of Hausdorff [Fund. Math. **25** (1935), 578] by showing that there is no real-valued function symmetrically continuous everywhere and discontinuous at precisely the points of Cantor's ternary set. *M. M. Day* (Seattle, Wash.).

**Mirkil, H. Differentiable functions, formal power series, and moments.** Proc. Amer. Math. Soc. **7** (1956), 650–652.

The theorem of E. Borel [Ann. Sci. Ecole Norm. Sup. (3) **12** (1895), 9–55, pp. 35–44] stating that for an infinitely-differentiable function of one variable $x$ the derivatives at $x_0$ can be arbitrarily assigned, is generalized by the author to the $n$-dimensional case in the following form: Let $\lambda_{p_1,\cdots,p_n}$ be an arbitrary family of complex numbers, indexed by $n$-tuples $(p_1, \cdots, p_n)$ of non-negative integers, then there exists some infinitely-differentiable function $f$ of $n$ real numbers that has as its derivatives at the origin exactly these complex numbers. The simple proof given is new even for the case $n=1$. Then the following proposition for moments is derived: Given an arbitrary family $\lambda_{p_1,\cdots,p_n}$ of complex numbers, indexed by $n$-tuples of non-negative integers, there exists an analytic function $f$ whose $(p_1, \cdots, p_n)$th moment is $\lambda_{p_1,\cdots,p_n}$ for each $(p_1, \cdots, p_n)$. *A. Rosenthal.*

**Alexievici, V. Geometric interpretation of differentials of higher order.** Gaz. Mat. Fiz. Ser. A. **8** (1956), 144–147. (Romanian)

Taylor's formula is illustrated by drawing osculating parabolas of different order. *O. Bottema* (Delft).

**Novotný, Miroslav. On representation of partially ordered sets by sequences of zeros and ones.** Časopis Pěst. Mat. **78** (1953), 61–64. (Czech)

Let $S$ be a partly ordered set. An $x \in S$ is said to have the $\alpha$-property or to be an $\alpha$-point of $S$ if $x$ is no initial and no terminal point of $S$ and if there exists a point $y \in S$ such that $y$ non $\leq x$ and that $t>x$ for each $t \in S$ satisfying $t>x$. Dually one defines the $\alpha^*$-property of an $x \in S$. A subset $H$ of $P$ is quoted as "dense in $S$" {we would say $N$-dense in $P$}, provided 1) $H$ contains every initial point of $S$ as well as every $\alpha$-point of $S$; and dually: $H$ contains every terminal point of $S$ as well as every $\alpha^*$-point of $S$; 2) if $a, b \in S$, $a<b$ then there exists $a', b' \in H$ such that $a\leq a'\leq b'\leq b$. The separability degree of $S$ is the least cardinal $m$ such that $S$ contains a set $H$ of cardinal $m$ dense in $S$. For any ordinal $\nu$ every ordered set $S$ of separability degree $\aleph_\nu$ is isomorph with a set of $\omega_\nu$-se-

quences of 0 and 1 ordered cardinally, i.e. so that $\{a_n\}\leq\{b_n\}$ if and only if $a_n\leq b_n$ for every $n<\omega_\nu$ (Th. 2).

*Đ. Kurepa* (Zagreb).

**Novotný, Miroslav. On similarity of ordered continua of types $\tau$ and $\tau^2$.** Časopis Pěst. Mat. 78 (1953), 59–60. (Czech)

No limited ordered chain $C$ is similar to an ordered system of closed intervals of $C$ (Th. 1); here the word "limited" is indispensable (Th. 3). No totally ordered set $X$ is similar to $X^\nu$, $\nu$ being any ordinal $>1$; $X^\nu$ denoting the alphabetically ordered set of all the $\omega_\nu$-sequences of elements of $X$ (Th. 2). Theorem 2 is the answer to a question raised by Novak. In connection with Th. 2 cf. Th. D p. 131 in Hausdorff, Ber. Verh. Sächs. Ges. Wiss. Leipzig. Math.-Phys. Kl. 58 (1906), 106–169.

*Đ. Kurepa* (Zagreb).

**Liverman, T. P. G. Implicit function theorem for quasi-analytic and related classes of functions.** Proc. Nat. Acad. Sci. U.S.A. 42 (1956), 261–264.

Soit $C^\infty$ l'ensemble de toutes les fonctions réelles, indéfiniment dérivables sur $[a, b]$; posons $\|f\|=\max|f(x)|+\max|f'(x)|$ $(x\in[a, b])$. $C^\infty$, muni de cette norme, sera désigné par $C_{(2)}{}^\infty$. Si $f\in C_{(2)}{}^\infty$ admet un zéro simple $\zeta=S(f)\in(a, b)$, il existe un voisinage $V_\varepsilon$ de $f$ dans lequel la fonctionnelle $S$ est continue et admet une différentielle de Gâteaux de tout ordre. $S(g)$ admet une série de Taylor. Des résultats plus précis sont énoncés pour les espaces $C_q\{M_n\}$ définis par $\|f\|=\sup[\max|f^{(n)}(x)|/q^nM_n]<\infty$ lorsque $f\in C\{M_n\}$.

*S. Mandelbrojt* (Paris).

**Gillman, L. Some remarks on $\eta_\alpha$-sets.** Fund. Math. 43 (1956), 77–82.

Let $\alpha$ be an ordinal number. Define $H_\alpha$ to be the lexicographically ordered set of all sequences $x=(x_\xi)_{\xi<\omega_\alpha}$ such that each $x_\xi$ is either 0 or 1, $x_{\varphi(x)}=1$ for some ordinal number $\varphi(x)<\omega_\alpha$ and $x_\xi=0$ for $\varphi(x)<\xi<\omega_\alpha$. Then every $\eta_\alpha$-set has a subset similar to $H_\alpha$. Let $\aleph_{\mu_\alpha}$ denote the smallest cardinal number $\mathfrak{p}$ such that there exists an $\eta_\alpha$-set of power $\mathfrak{p}$, and set $\nu_\alpha=\alpha$ if $\aleph_\alpha$ is regular, $\nu_\alpha=\alpha+1$ if $\aleph_\alpha$ is singular. Then, if $\alpha$ and $\delta$ are such that $\delta\geq\mu_\alpha$ and $\delta>\nu_\alpha$, there exist two dissimilar $\eta_\alpha$-sets of power $\aleph_\delta$. Suppose that $\alpha$ is a limit number. Then every ordered set of power $\aleph_\alpha$ is similar to a subset of $H_\alpha$ [for $\alpha$ isolated, cf. Sierpiński, Fund. Math. 36 (1949), 56–67; MR 11, 165]; $H_\alpha$ is an $\eta_\alpha$-set if, and only if, $\aleph_\alpha$ is regular; an $\eta_\alpha$-set of power $\aleph_\alpha$ exists if, and only if, $\aleph_\alpha$ is regular and $2^{\aleph_\beta}\leq\aleph_\alpha$ for every $\beta<\alpha$, and if this condition is satisfied, $H_\alpha$ is an $\eta_\alpha$-set of power $\aleph_\alpha$.

*F. Bagemihl* (Notre Dame, Ind.).

**Fodor, G. On a problem in set theory.** Publ. Math. Debrecen 4 (1956), 376–378.

Let $E$ be an uncountable set of cardinal $m$, let $n$ be a cardinal less than $m$, and let $R$ be a binary relation in $E$ such that for each $x\in E$ the set $\{y:y\in E, y\neq x, xRy\}$ has cardinal less than $n$. A subset of $E$ is termed free if for no distinct members $x$ and $y$ of this subset does $xRy$ hold. For the case where $m$ is singular and for each $x\in E$ the set $\{y:y\in E, yRx\}$ has cardinal less than $m$, it is shown (using transfinite induction) that $E$ must have a free subset of cardinal $m$. This answers a question of Ruziewicz for a case previously settled only under the generalized continuum hypothesis.

*T. A. Botts.*

**See also:** Ackermann, p. 3; Sprinkle, p. 3; Gładysz, p. 24; Marcus, p. 25; Mickle, p. 24; Krylov, p. 32; Tricomi, p. 36; Iséki and Miyanaga, p. 55.

## *Measure, Integration*

**Gładysz, S. Ein ergodischer Satz.** Studia Math. 15 (1956), 148–157.

**Gładysz, S. Über den stochastischen Ergodensatz.** Studia Math. 15 (1956), 158–173.

These two papers contain the detailed discussions and proofs of the facts the author announced earlier in Bull. Acad. Polon. Sci. Cl. III. 2 (1954), 411–413 [MR 16, 682].

*P. R. Halmos* (Chicago, Ill.).

**Besicovitch, A. S. On density of perfect sets.** J. London Math. Soc. 31 (1956), 48–53.

Given a bounded linear perfect set $E$, let $\{a_n\}$ be the sequence of interior complementary intervals in order of decreasing length. The parameter $\alpha$ is defined as $\liminf\alpha_n$, where the sequence $\{\alpha_n\}$ is defined by $r_n=\sum_{m\geq n}a_m$, $n(r_n/n)^{\alpha_n}=1$ ($r_n$ denoting both the set and its length). It is known that $\alpha\leq\delta$, where $\delta$ is the exponent of convergence of the series $\sum a_n$. In the present paper the author proves that the set of points of $E$ at which the upper density of $E$ is $<1$ is of Hausdorff dimension $\leq\alpha$, and the set of points of $E$ at which the lower density is $<1$ is of dimension $\leq\delta$.

*L. H. Loomis.*

**Mickle, Earl J. On the definition of significant multiplicity for continuous transformations.** Trans. Amer. Math. Soc. 82 (1956), 440–451.

Let $T:Q\to S_3$ be a continuous mapping from the unit square $Q:0\leq u\leq 1, 0\leq v\leq 1$, into Euclidean 3-space $S_3$, and let $A(T)$ denote the Lebesgue area of the Fréchet surface defined by $T$ (for concepts relating to surface area theory the reader is referred to the reviewer's book "Length and area" (to be quoted as LA) [Amer. Math. Soc. Colloq. Publ., vol. 30, New York, 1948; MR 9, 505]. The present paper constitutes an important contribution to the problem of defining in $S_3$ a multiplicity function $k(x, T)$, which depends upon both the point $x\in S$ and the mapping $T$, such that one has the formula $A(T)=\int k(x, T)dH^2$ for every $T$, where $H^2$ is two-dimensional Hausdorff measure in $S_3$. In a previous paper [Rend. Circ. Mat. Palermo (2) 4 (1955), 205–218; MR 17, 595] the author developed a novel idea for constructing such a multiplicity function, and in a subsequent paper [ibid. 4 (1955), 219–236; MR 17, 595] the reviewer simplified and extended the method of construction of Mickle. The multiplicity functions $k(x, T)$ exhibited in these papers depend however upon the choice of the coordinate system is $S_3$. The significant advance accomplished in the present paper is the construction of a multiplicity function $k(x, T)$ which is independent of the choice of the coordinate system. The following definitions are needed to describe this new construction. $U$ denotes the unit sphere $x_1^2+x_2^2+x_3^2=1$ in $S_3$. For $P\in U$, $S_2(P)$ is the plane through the origin which is perpendicular to the line joining $P$ to the origin. Points in $S_2(P)$ are denoted by $x_P$, and $\pi_P:S_3\to S_2(P)$ is the orthogonal projection from $S_3$ onto $S_2(P)$. The open sphere with center $x\in S_3$ and radius $r>0$ is denoted by $s(x, r)$. For plane sets, $L_2$ denotes two-dimensional Lebesgue measure. For $P\in U$, $\Gamma_P$ denotes the family of

those subsets of $S_3$ which are $H^2$-measurable and whose orthogonal projection upon $S_2(P)$ has $L_2$-measure zero. $H_P$ denotes the measure defined for $H^2$-measurable sets $E$ by the formula $H_P(E)=\inf H^2(E-E_P)$, $E_P \in \Gamma_P$. Let $P$ be a point of $U$, let $E$ be an $H^2$-measurable set in $S_3$, and let $n$, $m$ be positive integers. Then $G_{n,m}(E, P)$ is the set of those points $x \in S_3$ for which $H_P[E \frown s(x, r)] > \pi r^2/n$ for some $r$ such that $0<r<1/m$. For each continuous mapping $T$ (see above) and for each Borel set $B$ in the $uv$-plane the set $D^*(T, B)$ is defined by the formula

$$D^*(T, B)=\bigcup_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{P \in U} G_{m,m}\{T[B \frown E_P(T)], P\},$$

where $E_P(T)$ is the union of all the essential maximal model continua (see LA, pp. 281–282) under the mapping $\pi_P T: Q \to S_2(P)$. Let $\Omega$ be the class of all the open sets in the $uv$-plane. Given the mapping $T$ as above, a maximal model continuum $C$ under $T$ is termed significant under $T$ if $T(C) \in D^*(T, O)$ for every set $O$ such that $C \subset O \in \Omega$. The union of all the significant maximal model continua under $T$ is denoted by $S(T)$. The new multiplicity function $k(x, T)$ of Mickle is then defined, for $x \in S_3$, as the number (possibly infinite) of those maximal model continua under $T$ that intersect the set $T^{-1}(x) \frown S(T)$. Then $k(x, T)$ is obviously independent of the choice of a coordinate system in $S_3$, and by an argument analogous to that in his paper quoted above the author verifies the formula $A(T)=\int k(x, T)dH^2$. *T. Radó* (Columbus, Ohio).

**Fell, J. M. G. A note on abstract measure.** Pacific J. Math. 6 (1956), 43–45.

Let the dimension of a measure space be defined as the least cardinal number of a collection of measurable sets of finite measure that is maximal with respect to the property that any two members intersect in a null set, and suppose that any subset of a null set is measurable. Then if the dimension is not greater than the smallest uncountable cardinal, there exists a disjoint family of measurable sets of finite measure, such that the measure of any measurable set of finite measure is the sum of the measures of its intersections with the members of the family. In particular, under the assumption of the continuum hypothesis, the conclusion of the Radon-Nikodym theorem is valid for the space. *I. E. Segal.*

**Prékopa, A.; Rényi, A.; and Urbanik, K. On the limiting distribution of sums of independent random variables in bicompact commutative topological groups.** Acta Math. Acad. Sci. Hungar. 7 (1956), 11–16. (Russian. English summary)

The main result is the following: If $\xi_n$ ($n=1, 2, \cdots$) are independent equidistributed random variables assuming values in a compact commutative topological group $G$ and if $P(\xi_n \in U)>0$ for every open non-empty subset $U$ of $G$, then the sequence of measures

$$\mu_n(E)=P(\xi_1+\cdots+\xi_n \in E),$$

defined for all Borel subsets of $G$, converges weakly to the Haar probability measure on $G$. The proof is based on reducing the theorem, through consideration of the character group, to the special cases when $G$ is either the additive group of reals mod 1 or a finite cyclic group, which were proved by P. Lévy [Bull. Soc. Math. France 67 (1939), 1–41; MR 1, 62] and [among others, see below] A. Dvoretzky and J. Wolfowitz [Duke Math. J. 18 (1951), 501–507; MR 12, 839].

{Remark by reviewer: It seems to be little known that the above result even for non-commutative compact groups, as well as many related results, were proved by Y. Kawada and K. Itô [Proc. Phys.-Math. Soc. Japan (3) 22 (1940), 977–998; MR 2, 223] by straightforward use of the theory of unitary representations. Kawada and Itô make the additional assumption that the group is separable, but developments since 1940 have made it well-known that this condition is superfluous.}

*A. Dvoretzky* (New York, N.Y.).

**Albuquerque, J. Une théorie de la mesure des ensembles, au sens de Lebesgue, dans les espaces abstraits.** Univ. Lisboa. Revista Fac. Ci. (2) 5 (1955–1956), 147–168.

Dans cet article l'auteur étudie les mesures extérieures (m.e.) définies sur l'ensemble des parties $\mathcal{P}(E)$ d'un ensemble $E$. Après avoir rappelé certains théorèmes classiques, il suppose $\mathcal{P}(E)$ muni d'une application $A \to \bar{A}$ telle que $\emptyset=\bar{\emptyset}$, $A \subset \bar{A}$, $A \subset B$ implique $\bar{A} \subset \bar{B}$, $\bar{\bar{A}}=\bar{A}$ et donne quelques résultats sur les m.e. $\mu$ vérifiant la relation (1) $\mu(A \cup B)=\mu(A)+\mu(B)$ si $A \frown \bar{B}=\emptyset$ et sur les m.e. $\mu$ vérifiant, en plus, la relation (2) $\mu(A)=\mu(\bar{A})$. {Remarques du rapporteur: Pour obtenir une m.e. par la méthode de prolongement, indiquée au commencement du § 5, il faut introduire certaines hypothèses; par exemple, supposer que $\overline{\bigcup_{i=1}^{\infty} A_i}=\bigcup_{i=1}^{\infty} \bar{A}_i$. La dénomination, m.e. de Lebesgue, donnée par l'auteur aux m.e. satisfaisant aux conditions (1) et (2) n'est pas adéquate.}

*C. T. Ionescu Tulcea* (Bucarest).

**Marcus, S. Sur un problème de la théorie de la mesure de H. Steinhaus et S. Ruziewicz.** Bull. Acad. Polon. Sci. Cl. III. 4 (1956), 197–199.

A linear set $E$ is totally asymmetric if, for every $x \in E$, there is an $h(x)>0$ such that, whenever $|h|<h(x)$, either $x-h \notin E$ or $x+h \notin E$. It is shown that if $E$ is totally asymmetric it is of interior measure 0; if, in addition, it has the property of Baire, it is of the first category.

*C. Goffman* (Norman, Okla.).

See also: Šanin, p. 2.

## *Functions of Complex Variables*

**Reade, Maxwell O. A sufficient condition that $f(z)$ be analytic.** Arch. Math. 7 (1956), 126–128.

Let $\{C_n\}$ be a system of analytic Jordan curves with interiors $\{D_n\}$, and let $\{C_n\}$ be a null-sequence in the sense of Müller [Arch. Math. 6 (1954), 47–51; MR 16, 346]. The sufficient condition alluded to in the title, that a function $f(z)$, continuous in $|z|<1$, be analytic in $|z|<1$, is that (1) there exists a null-sequence $\{C_n\}$ with the properties

$$\iint_{D_n} \zeta d\xi d\eta=\iint_{D_n} \zeta^2 d\xi d\eta=0,$$

$$\frac{A_n^2}{a} \leq \iint_{D_n} |\zeta|^2 d\xi d\eta < aA_n^2 \quad (\zeta=\xi+i\eta)$$

for some $a>0$, where $A_n$ is the area of $D_n$, and (2) for every convergent sequence $\{z_n\}$ in $|z|<1$,

$$\lim_{n\to\infty} A_n^{-2} \iint_{D_n} \zeta f(z_n+\zeta) d\xi d\eta=0.$$

*A. J. Lohwater* (Ann Arbor, Mich.).

**Lohwater, A. J. The boundary behavior of a quasi-conformal mapping.** J. Rational Mech. Anal. 5 (1956), 335–342.

Beurling has shown that if $f(z)$ is analytic and univalent in $|z|\leq 1$, then $f$ has non-tangential limits on $|z|=1$ except possibly on a set of capacity zero. In the present paper the author gives a proof of this theorem in the case of a quasi-conformal mapping $f$. His proof is based on a proof by Tsuji for the conformal case. *H. L. Royden.*

**Jenkins, James A. On quasiconformal mappings.** J. Rational Mech. Anal. 5 (1956), 343–352.

The author proves the same theorem as Lohwater in the paper reviewed above, but does so by first proving: A) that arcs approaching $|z|=1$ non-tangentially are carried by a quasi-conformal mapping into arcs which approach $|z|=1$ non-tangentially, and B) that sets of capacity zero are carried into sets of capacity zero by such a mapping. Thus the principal theorem is reduced to Beurling's theorem by the use of an auxiliary conformal map.

He also shows that the generalizations to quasi-analytic functions of the Fatou theorem (on the existence of radial limits for bounded analytic functions) and the Lusin-Privaloff theorem (stating the constancy of an analytic function constant on a set of positive measure on $|z|=1$) are both equivalent to the statement that a quasi-conformal mapping of $|z|<1$ into itself takes sets of linear measure zero on $|z|=1$ into set of linear measure zero. *H. L. Royden* (Stanford, Calif.).

**Protter, M. H. The periodicity problem for pseudoanalytic functions.** Ann. of Math. (2) 64 (1956), 154–174.

This paper solves an open question in the theory of pseudoanalytic functions. Let $F(z)$, $G(z)$ be two complex-valued Hölder continuous functions defined in some domain such that $\mathrm{Im}(\bar{F}G)>0$. A function $w=\phi F+\psi G$, where $\phi$ and $\psi$ are real, is called $(F, G)$ pseudoanalytic if $\phi_{\bar{z}}F+\psi_{\bar{z}}G=0$. The function $\dot{w}=\phi_z F+\psi_z G$ is called the $(F, G)$ derivatives of $w$. Every generating pair $(F, G)$ has a successor $(F_1, G_1)$ such that $(F, G)$ derivatives are $(F_1, G_1)$ pseudoanalytic. The successor is not uniquely determined. A generating pair $(F, G)$ is said to have minimum period $n$ if there exists generating pairs $(F_i, G_i)$ such that $(F_0, G_0)=(F, G)$, $(F_{i+1}, G_{i+1})$ is a successor of $(F_i, G_i)$, and $(F_n, G_n)=(F_0, G_0)$. If such an $n$ does not exist, $(F, G)$ is said to have minimum period $\infty$. It is easy to give examples of generating pairs having minimum periods 1 and 2. It was not known whether there exist generating pairs with given minimum periods and whether there exist generating pairs with minimum period $\infty$. The present article gives an affirmative answer to both questions, and this solves the periodicity problem which was first stated, in the framework of a different formalism, by Markushevitch [cf. Petrovskii, Uspehi Mat. Nauk. (N.S.) 1 (1946), no. 3-4(13-14), 44–70; MR 10, 301; 11, 520].

The author considers real analytic generating pairs. These are known to have only real analytic successors. A generating pair $(F, G)$ has characteristic coefficients $a, b, A, B$ defined by

$$F_{\bar{z}}=aF+b\bar{F},\quad G_{\bar{z}}=aG+b\bar{G}\ ,$$
$$F_z=AF+B\bar{F},\ G_z=AG+B\bar{G}.$$

$(F_1, G_1)$ is a successor of $(F, G)$ if, and only if, $a_1=a$, $b_1=-B$. The author proves that given functions $a, b, A, B$ are characteristic coefficients of a generating pair if and only if they satisfy the differential equations $A_{\bar{z}}=a_z+b\bar{b}-B\bar{B}$, $B_{\bar{z}}=b_z+(\bar{a}-A)b+(a-\bar{A})B$. Using this, he derives an overdetermined system of differential equations for the characteristic coefficients of a generating pair having minimum period $n$. He then shows how to construct power series which satisfy these equations for a given $n$ and for no smaller number, or for no $n$. Thus the nature of the proof is elementary but the actual procedure is very involved.

The author raises the interesting open question whether any successor of a generating pair can have a smaller minimum period than the original pair. *L. Bers.*

**Herzog, Fritz; and Piranian, George. Some properties of the Fejér polynomials.** Proc. Amer. Math. Soc. 7 (1956), 379–386.

Die für verschiedene Untersuchungen der Fourier- und Potenzreihen nützlichen Fejérschen Polynome

$$P_n(z)=\frac{1}{n}+\frac{z}{n-1}+\cdots+\frac{z^{n-1}}{1}-\frac{z^n}{1}-\frac{z^{n+1}}{2}-\cdots-\frac{z^{2n-1}}{n}$$
$$(n=1, 2, \cdots)$$

werden hier auf Lage der Nullstellen und Maximum $M_n=\max_{|z|=1}|P_n(z)|$ hin untersucht. Die Nullstellen häufen sich gegen $|z|=1$ und lassen sich außerdem in Sektoren von sehr engen Kreisringen einschließen. Während $P_n(z)$ für ungerades $n$ keine negativen Wurzeln hat, besitzt es für gerades $n$ zwei negative Wurzeln an den Stellen

$$z=-1\pm\left(\frac{\log n}{n}+\frac{\log\log 16}{n}\right)+o\left(\frac{1}{n}\right).$$

Für $M_n$ wird $M_n\to M=2\int_0^\pi \sin t/t\,dt=3{,}704\cdots$ $(n\to\infty)$ bewiesen und ferner erwähnt, daß sogar stets $M_n\leq M$ gilt. *D. Gaier* (Stuttgart).

**Piranian, George. The orders of lacunarity of a power series.** Boll. Un. Mat. Ital. (3) 11 (1956), 198–199.

Diese Note schließt an eine Arbeit von Ricci an [Rend. Mat. e Appl. (5) 14 (1955), 602–632; MR 17, 598]. Dort wurde unter anderem bewiesen (Bezeichnungen des genannten Referates): Sind $\Lambda$, $\Lambda^*$ zwei beliebige Werte des Intervalls $[0, \infty]$, so gibt es eine Potenzreihe $\sum a_n z^n$ mit $\limsup |a_n|^{1/n}=1$ und $\Lambda$, $\Lambda^*$ als 'orders of lacunarity'. Der Verf. gibt dafür ein einfaches und durchsichtiges Beispiel. *D. Gaier* (Stuttgart).

**Mayer-Kalkschmidt, Jörg. Über Singularitäten gewisser Potenzreihen.** Arch. Math. 7 (1956), 129–134.

On entend par la dérivée de gauche d'ordre $(\nu+1)$ de (*) $f(z)=\sum_1^\infty a_\nu z^\nu$ en $z=1$, $\lim_{z\to 1-0}\{f^{(\nu)}(z)-f^{(\nu)}(1)\}(z-1)^{-1}=f_L^{(\nu)}(1)$, $\nu=0, 1, \cdots$, $f_0(z)=f(z)$. Alors, l'auteur a démontré que la fonction $f(z)$ avec le cercle de convergence $|z|=1$, et les $a_\nu$ réels, possède une singularité en $z=1$ si dans ce point, les moyennes arithmétiques des sommes partielles de (*) d'ordre $k>0$ forment une suite croissante et si $f_L^{(\nu)}(1)=0$, $\nu=1, 2, \cdots, k$. *M. Tomić* (Beograd).

**Mitrović, Dragiša. Sur les valeurs de certaines intégrales définies.** Hrvatsko Prirod. Društvo. Glasnik Mat.-Fiz. Ser. II. 10 (1955), 259–263. (Serbo-Croatian summary)

Let $D$ be the interior of a simple closed curve $C$. Let $h(z)$ be a function regular in the closure of $D$ and suppose that a second function $f(z)$ be likewise regular in this closure, except for a finite number of isolated singularities

in $D$. We suppose that $f(z)$ does not vanish in $D$ and on $C$. The author, using the classical theory of residues, obtains formulae for the computation of the integrals

$$\int_C h(z)d\log f(z),\ \int_C h'(z)\log f(z)dz$$

in terms of the coefficients of the Laurent expansions of $f'(z)/f(z)$ in annuli about the singular points.

Following the publication of his paper, the author has informed us of an error appearing in formula (21), page 261. The formula may be corrected if the symbol Re (real part of) is moved back to precede the double summation sign to allow for the possibility that the coefficients $B_{n,\nu}$ may not be real. In addition the condition should be added that $a_\nu \neq 0$, $\nu=1, 2, \cdots, k$ in this formula.

*M. S. Robertson* (New Brunswick, N.J.).

**Kuramochi, Zenjiro. Capacity of subsets of the ideal boundary.** Proc. Japan Acad. 32 (1956), 111–116.

The author gives simpler proofs of the theorems in same Proc. **30** (1954), 951–956; **31** (1955), 25–30 [MR **16**, 1013].

*H. L. Royden* (Stanford, Calif.).

**Johnson, Guy, Jr. Collective singularities of a family of analytic functions.** Proc. Amer. Math. Soc. **7** (1956), 653–655.

Let $\mathcal{F}$ be a family of functions $f(z)=\sum a_n(f)z^n$. A set $D$ is a domain of regularity of $\mathcal{F}$ if every $f$ is holomorphic and the family is normal in $D$. A point $z'$ on the boundary of $D$ is a regular point of $\mathcal{F}$ if there exists some neigborhood $U$, of $z'$, such that $U+D$ is a domain of regularity. Otherwise $z'$ is said to be a singular point of $\mathcal{F}$. Assume (i) that 1 is the radius of the greatest circle, with center at the origin, in which every $f$ is holomorphic and $\mathcal{F}$ is normal; (ii) $\mathcal{F}$ is uniformly bounded on each compact subset of $|z|<1$; (iii) to each $f$ there corresponds a sequence of integers $\{\lambda_p(f)\}_{p=1}^\infty$ such that $a_n(f)=0$ if $n\neq\lambda_p(f)$, $(p=1, 2, 3, \cdots)$; moreover the gaps are uniform in the sense: $\lambda_{p+1}(f)/\lambda_p(f)\geq\lambda>1$, where $\lambda$ is independent of $f$. Then each point of $|z|=1$ is a singular point of $\mathcal{F}$.

*A. Edrei* (Syracuse, N.Y.).

**Roşculeţ, Marcel N. Théorie des fonctions de variable hypercomplexe dans l'espace à trois dimensions.** Acad. R. P. Române. Bul. Şti. A. 1 (1949), 523–528. (Romanian. Russian and French summaries)

The author extends the theory of analytic functions to the linear algebra over the real field whose basis elements are 1, $\theta$, $\theta^2$ ($\theta^3=1$). The conditions for monogenity and integrability derived for this algebra are special cases of the theory given for large classes of linear algebras by P. W. Ketchum [Trans. Amer. Math. Soc. **30** (1928), 641–667], J. A. Ward [Duke Math. J. **7** (1940), 233–248; MR **2**, 122], and R. D. Wagner [ibid. **15** (1948), 455–461; MR **10**, 30].

*J. A. Ward* (Holloman, N.M.).

**Johnson, Guy, Jr. Regions of flatness for analytic functions and their derivatives.** Duke Math. J. **23** (1956), 209–217.

Soit $S$ un ensemble de domaines simplement connexes $D$, dont aucun n'est le plan entier. Soit $\omega$ la fonction holomorphe, univalente, transformant $|w|<1$ en $D$, avec $\omega(0)=a\in D$, $\omega'(0)>0$. Désignons par $D(h)$ la transformée, par $\omega$, de $|w|<h$ $(0<h<1)$. Soit $f(z)$ une fonction holomorphe dans un domaine contenant les domaines de $S$, et supposons que $f(z)\neq 0$ dans $D(h)$, sauf pour un nombre fini de $D$ (de $S$, ce nombre dépendant de $h$). Si la famille $\mathcal{F}$ des fonctions $F(w)$ définies par $F(w)=f[\omega(w)]$ est normale dans $|w|<1$, les domaines $D$ de $S$ constituent un ensemble de plaines pour $f$ ("regions of flatness"). L'auteur indique des conditions suffisantes pour qu'un ensemble de plaines pour $f$ soit aussi un ensemble de plaines pour $f'$. Des conditions sont aussi indiquées pour que ces régions constituent un ensemble de plaines pour toutes les dérivées de $f$.

*S. Mandelbrojt* (Paris).

**Mori, Akira. On an absolute constant in the theory of quasi-conformal mappings.** J. Math. Soc. Japan **8** (1956), 156–166.

Let $T$ be a quasi-conformal mapping of $|z|\leq 1$ onto $|Tz|\leq 1$ with maximal dilatation $\leq K$. The author proves that $|Tz_1-Tz_2|\leq 16|z_1-z_2|^{1/K}$ and that 16 is the best constant independent of $K$. The proof makes use of an interesting lemma: Among all doubly connected regions with the property that one complementary component contains 0 and $\infty$ while the other intersects $|z|<1$ in a set with given diameter $\lambda<2$, the one with maximal modulus is obtained by deleting from the plane the negative real axis and an arc on $|z|=1$, symmetrically to the real axis.

*L. Ahlfors* (Cambridge, Mass.).

**Hiong, King-Lai. Sur l'impossibilité de quelques relations identiques entre des fonctions entières.** C. R. Acad. Sci. Paris **243** (1956), 222–225.

Proof by elementary methods that every nonconstant integral function $F(z)$ satisfies either $F(z)=0$ or $F'(z)=1$ somewhere. A proof of this by Nevanlinna Theory methods is due to Milloux [Les fonctions méromorphes et leurs dérivées···, Hermann, Paris, 1940; MR **7**, 427]. The author also gives some extensions being careful not to transgress beyond the ideas of Milloux.

*W. K. Hayman* (London).

**Rudin, Walter. The radial variation of analytic functions.** Duke Math. J. 22 (1955), 235–242.

Let $U$ denote the circle $|z|<1$ and $K$ its closure. Let $f(z)$ be a function analytic in $U$ and

$$V(f,\theta)=\int_0^1|f'(re^{i\theta})|dr.$$

After stating some known positive results about the finiteness of $V(f,\theta)$ and proving that if

$$f(z)=\prod_1^\infty\frac{b_n-z}{1-\bar{b}_nz}\frac{|b_n|}{b_n}$$

with $|b_n|<1$, $\sum a_n<\infty$, $\sum a_n\log a_n^{-1}<\infty$, where $a_n=1-|b_n|$, then $V(f,\theta)<\infty$ for almost all $\theta$, the author shows that there exist a bounded function, a Blaschke product and an $f$ analytic in $U$ and continuous in $K$ for which $V(f,\theta)=\infty$ except possibly on a set of the first category of measure zero.

*A. P. Calderón* (Cambridge, Mass.).

**af Hällström, Gunnar. Zur Berechnung der Bodenordnung oder Bodenhyperordnung eindeutiger Funktionen.** Ann. Acad. Sci. Fenn. Ser. A. I. no. **193** (1955), 16 pp.

Let $E$ be a closed set of capacity zero and $B$ its complement with respect to the whole $z$-plane. H. Selberg [Avh. Norske Vid.-Akad. Oslo. I. **1937**, no. 10] proved that there exists a function $g(z)$ such that $g(z)$ is harmonic in $B-(z_0)$, $g(z)=\ln|z-z_0|+$regular harmonic function in a neighborhood of $z_0\in B$ and the boundary value of $g(z)$ at every point of $E$ is $+\infty$. Such a function $g(z)$ is called a Selberg function. If $E$ consists of a single point, $g(z)$ is uniquely determined except for an additive constant. But

the same situation does not hold if $E$ contains at least two points. Now, suppose that $f(z)$ is a single-valued meromorphic function in $B$. To extend Nevanlinna's theory of meromorphic functions to the case of general domains, the author introduced the characteristic function $T(\lambda)$ of $f(z)$ with the aid of the level curve: $g(z)=\lambda$ (instead of $|z|=r$) and defined the order of $f(z)$ by $s=\limsup \ln T(\lambda)/\lambda$, in the case $s=\infty$, the hyperorder of $f(z)$ by $S=\limsup \ln \ln T(\lambda)/\lambda$ [cf. Acta Acad. Abo. **12** (1939), no. 8; MR **2**, 275]. The order (hyperorder) depends upon the choice of $g(z)$. To obtain a fixed concept of order, the author introduces the basic order (Bodenordnung) $s_0$, and the basic hyperorder (Bodenhyperordnung) $S_0$ by the definitions: $s_0=\inf s$, $S_0=\inf S$ respectively, considering the sets of values of $s$ and $S$ which are obtained for various Selberg functions $g(z)$ [cf. Proc. Intern. Congr. Math. Amsterdam, (1954), II, Noordhoff, Groningen, 1954, p. 117]. In the present paper, the author first proves a lemma on the potential-representation of an arbitrary Selberg function $g(z)$. Then follow three examples to explain the concepts of basic order and basic hyperorder. Through these examples, the author discusses whether the basic order (hyperorder) is attained by a certain $g(z)$ or not. In these three examples, $E$ consists, first, of $n$ points, secondly, of a sequence of points tending to infinity and thirdly of a generalized Cantor set. Furthermore, the author gives a new proof for a criterion of Nevanlinna which decides whether a Cantor set is of capacity zero or not. *K. Noshiro* (Nagoya).

**Riiber, Ågot E. Über meromorphe Funktionen mit einem Existenzgebiete, dessen Rand eine Cantor'sche Punktmenge von der Kapazität null ist.** Math. Scand. **3** (1955), 229–242 (1956).

Hällström has extended (see paper reviewed above) the theory of Nevanlinna to single-valued functions $f(z)$ which are meromorphic in a given multiply connected domain $G$. The present paper deals with the case when the boundary points of $G$ form a generalized Cantor set of capacity zero. With the aid of the Green's function (Evans-Selberg's function) of $G$ the characteristic function of Nevanlinna is defined, its relation to the distribution of values of $f(z)$ in $G$ studied and the defect-relation established.

*G. Szegö* (Stanford, Calif.).

**Hiong, King-Lai. Sur la croissance des fonctions algébroïdes en rapport avec leurs dérivées.** C. R. Acad. Sci. Paris **242** (1956), 3032–3035.

Various defect relations for algebroid functions.

*W. K. Hayman* (London).

**Lohwater, Arthur J. Sur le principe de symétrie et la répartition des valeurs des fonctions analytiques bornées.** C. R. Acad. Sci. Paris **242** (1956), 2278–2281.

Let $f(z)$ be regular and bounded in the unit circle $|z|<1$. If the radial limit $f(e^{i\theta})=\lim_{r\to 1} f(re^{i\theta})$ has absolute value 1 almost everywhere on $|z|=1$, then $f(z)$ is called a function of class $(U)$ in Seidel's sense. The purpose of this paper is to give a method to investigate systematically functions of class $(U)$ and some related classes, from the view-point of value-distribution. The author's method is of topological nature and based on the following lemma: Let $f(z)$ be a non-constant function of class $(U)$ such that $f(z)\neq 0$ in $|z|<1$. Then, for a fixed $n$, each (connected) component $G_{n,k}$ of the open set

$$H_n=\{z|\ |z|<1;\ |f(z)|<1/n\}\quad (n=2, 3, \cdots),$$

is simply connected and the intersection of the frontier Fr $G_{n,k}$ with $K$: $|z|=1$ is non-empty and of linear measure zero. Furthermore, each component $G_{n,k}$ $(k=1, 2, \cdots)$ contains at least one component of the set $H_{n+1}$.

*K. Noshiro* (Nagoya).

**Jenkins, J. A. On explicit bounds in Landau's theorem.** Canad. J. Math. **8** (1956), 423–425.

Let $F(z)=a_0+a_1z+\cdots$ be regular in $|z|<1$ and satisfy $F(z)\neq 0, 1$ there. The author proves

$$|a_1|\leq 2|a_0|\{|\log |a_0||+A\}$$

with $A=5.94$. He previously proved this result with $A=7.77$ [same J. **7** (1955), 76–82; MR **16**, 579]. With $A=4.37$ the result is no longer true. *W. K. Hayman.*

**Lewy, Hans. On the relations governing the boundary values of analytic functions of two complex variables.** Comm. Pure Appl. Math. **9** (1956), 295–297.

A continuous function $f(x_1, y_1)$ of the real variables $x_1, y_1$ is called $[m^0, c^0]$-continuously summable if

$$\int_{-\infty}^{\infty} |f(x_1, mx_1+c)|\, dx_1 = \int_{[m,c]} |f|\, dx_1$$

converges uniformly when $[m, c]$ belongs to a neighbourhood of $[m^0, c^0]$. Let $\bar{U}$ be the class of all real functions $\bar{u}$ bounded and continuous in the $x_1y_1$-plane and satisfying the conditions: i) $\int_{[m,c]} |\bar{u}|dx_1$ is bounded when $m$ is bounded away from zero and infinity. ii) $\partial\bar{u}/\partial y_1$ is $[m^0, c^0]$-continuously summable when $m^0>0$. Further, let $Z$ be the class of analytic functions of $x=x_1+ix_2$, $y=y_1+iy_2$ regular when $x_2>0$, $y_2>0$, tending to zero when $x_2$ tends to infinity while $y_2/x_2$ remains bounded away from zero and infinity. The author proves that a function $\bar{u}$ of the class $\bar{U}$ can be the boundary function on the real plane of a function of the class $Z$ if and only if the integral $\int_{[m,c]} |\bar{u}|dx_1$ for positive $m$ is independent of $c$.

*H. Tornehave* (Virum).

**Bremermann, H. J. Complex convexity.** Trans. Amer. Math. Soc. **82** (1956), 17–51.

On fait une étude de la pseudo-convexité des domaines dans $C^n$ au moyen des fonctions plurisousharmoniques en insistant sur l'analogie formelle entre fonctions convexes dans $R^p$ d'une part et fonctions plurisousharmoniques dans $C^n$ d'autre part; la rédaction complète les résultats de la thèse de l'auteur [Schr. Math. Inst. Univ. Münster no. **5** (1951); MR **14**, 971] et passe en revue des résultats connus [cf. P. Lelong, J. Math. Pures Appl. (9) **31** (1952), 191–219; J. Analyse Math. **2** (1952), 178–208; MR **14**, 463, 971]. On y ajoute le résultat intéressant suivant: si $N(z)$ est une norme sur $C^n$ considéré comme espace vectoriel, c'est-à-dire une fonction satisfaisant à $N(0)=0$, $N(z)>0$ pour $z\neq 0$, $N(z^1+z^2)\leq N(z^1)+N(z^2)$, $N(\lambda z)=|\lambda|N(z)$, alors un domaine $D$ pour lequel $-\log \delta(z)$ est plurisiousharmonique de $z\in D$, $\delta(z)$ étant la distance euclidienne de $z\in D$ à la frontière de $D$, possède aussi cette propriété si l'on substitue à $\delta(z)$ la distance $\delta_N(z)$ calculée à partir de $N(z)$. Si $\chi(z)$ est holomorphe dans $D$ et ne s'y annule pas, $|\chi(z)|\delta_N(z)$ satisfait au principe du maximum sur les domaines pris dans les plans à une dimension complexe. Dans la seconde partie de l'article, on s'appuie sur le fait que les fonctions plurisousharmoniques des $z_k=x_k+iy_k$ qui sont indépendantes des $y_k$ sont exactement les fonctions convexes de l'ensemble des $x_k$, pour mettre en parallèle des propriétés connues des fonctions plurisousharmoniques et des fonctions convexes.

*P. Lelong* (Princeton, N.J.).

**See also:** Pyateckiĭ-Šapiro, p. 19; Pisot, p. 19; Cowling, p. 31; Koutský p. 30; Cazenave, p. 33; Numerov, p. 91; Destouches, p. 95.

## *Functions with particular properties*

**Matsushita, Shin-Ichi. Sur la décomposition de F. Riesz. I, II.** C. R. Acad. Sci. Paris **241** (1955), 1252–1254, 1373–1375.

Démonstration en langage moderne du théorème de décomposition de F. Riesz relatif aux fonctions surharmoniques. {Certains points semblent obscurs au rapporteur.}

*J. Deny* (Strasbourg).

**Matsushita, Shin-ichi. Théorème de Krein-Milman et le balayage de mesures dans la théorie du potentiel. I, II, III.** Proc. Japan Acad. **31** (1955), 643–647; **32** (1956), 29–34, 125–130.

L'auteur donne une interprétation nouvelle et intéressante du balayage et des points irréguliers en théorie newtonienne. Je modifie un peu des notations, qui prêtent à confusion: Soit $L_\infty$ l'espace des fonctions réelles continues sur $R^n$ ($n\geq 3$) nulles à l'infini, muni de la norme uniforme; son dual $M$ est l'ensemble des mesures de Radon réelles de variation totale finie. On considère l'ensemble des potentiels newtonians continus, engendrés par des mesures à support compact dans un ouvert donné $\omega$ de $R^n$ et on désigne par $H$ la variété linéaire fermée de $L_\infty$ engendrée par ces potentiels. A toute $\mu \in M$ correspond un élément $\mu^\wedge$ du dual $H^\wedge$ de $H$: $\mu^\wedge(f)=\int f d\mu$ pour toute $f \in H$; on définit ainsi un homomorphisme $\mu \to \mu^\wedge$ de $M$ sur $H^\wedge$, continu pour les topologies faibles, et l'image de l'ensemble $M_1^+$ des mesures de Radon $\geq 0$, de masse totale $\leq 1$, est un ensemble convexe et compact $M_1^\wedge$ de $H^\wedge$.

Le résultat essentiel est que les points extrémaux de $M_1$ sont l'élément nul de $H^\wedge$ et les éléments de la forme $\varepsilon_x^\wedge$, où $\varepsilon_x$ désigne la masse $+1$ en un point $x$ intérieur à $\omega$ ou point-frontière régulier (point en lequel $\omega$ n'est pas effilé). D'autre part la mesure balayée sur $\omega$ d'une $\mu \in M_1^+$ est construite à partir du fait que $M_1^\wedge$ est l'enveloppe fermée convexe de l'ensemble de ses points extrémaux (Krein et Milman).

{Plusieurs inexactitudes sont à relever: par exemple, dans les parties I et II on définit et étudie en réalité non pas le balayage (d'une $\mu$ portée par le domaine $D$) sur la frontière $\Gamma$ de $\bar{D}$, mais le balayage sur l'ouvert $\omega$ complémentaire de $D$, ces deux notions pouvant être distinctes; et il n'est pas toujours vrai que l'ensemble des points-frontière irréguliers de $\omega$ soit de capacité nulle.}

*J. Deny* (Strasbourg).

**Tsuji, Masatsugu. On Dirichlet-and Neumann- problem with integrable boundary values.** Jap. J. Math. **23** (1953), 15–37 (1954).

Résolution des problèmes en question pour un domaine „très régulier" du plan ou de l'espace lorsque la donnée-frontière est une fonction sommable (pas nécessairement continue). L'utilisation des équations intégrales exige une étude préliminaire du comportement à la frontière des potentiels engendrés par une simple ou double couche de densité sommable. {Aucune mention n'est faite des travaux déjà anciens sur ce sujet, en particulier du mémoire de G. C. Evans et E. R. C. Miles [Amer. J. Math. **53** (1931), 493–516] dans lequel on considère le cas plus général où la donné-frontière est une mesure de Radon.}

*J. Deny* (Strasbourg).

**Duffin, R. J. Continuation of biharmonic functions by reflection.** Duke Math. J. **22** (1955), 313–324.

The author derives various formulas for continuation of biharmonic functions across flat or spherical parts of the boundary. The essential tool is the well-known representation of a biharmonic function by a pair of harmonic functions, which permits reducing each continuation problem on biharmonic functions to one on systems of harmonic functions. The proofs hold for arbitrarily high dimensions. By a Kelvin transformation the case of a spherical boundary can be reduced to that of planar boundary. Also continuation of solutions of the Navier-Stokes equations are treated in a similar fashion by representing the solutions in terms of 3 harmonic functions. The following theorem may serve as representative of the character of the paper.

Let $\mathcal{E}^*$ be an open set of the $x$-$y$-$z$-space and symmetric with respect to the plane $x=0$. Let $\theta$ be the intersection of $\mathcal{E}^*$ with this plane and $\mathcal{E}$ the intersection of $\mathcal{E}^*$ with the half-space $x>0$. Suppose that $w(x, y, z)$ is biharmonic in $\mathcal{E}$ and that $w/x$ has boundary values zero on $\theta$. Then $w$ can be continued into $\mathcal{E}^*$ by the formula

$$w(-x, y, z)=-w(x, y, z)+2x\partial w(x, y, z)/\partial x-x^2\Delta w(x,y,z).$$

The corresponding formula for two dimensions was derived by Poritsky [Trans. Amer. Math. Soc. **59** (1946), 248–279; MR **7**, 449] but under too restrictive conditions regarding the behavior of $w$ at the boundary $\theta$.

*C. Loewner* (Stanford, Calif.).

**Kishi, Masanori. On a theorem of Ugaheri.** Proc. Japan Acad. **32** (1956), 314–319.

In this note, the author generalizes a maximum principle due to Ugaheri [Bull. Tokyo Inst. Tech. **4** (1953), 149–179], for potentials in $n$-space, to potentials defined in a locally compact space $\Omega$. The author proves that the "continuity principle" plus the proposition "if the potential of a positive mass with compact carrier is bounded on that carrier, then it is bounded on $\Omega$", imply the new (generalized) "maximum principle." Conversely, the generalized "maximum principle" implies the "continuity principle."

*M. Reade* (Ann Arbor, Mich.).

**See also:** Liboff, p. 30.

## *Special Functions*

**Ananda-Rau, K. On certain infinite series for doubly periodic functions.** J. Indian Math. Soc. (N.S.) **19** (1955), 95–103 (1956).

A meromorphic function $f(u)$ of the complex variable $u$ is called quasi-elliptic with quasi-periods $2\omega$ and $2\omega'$ if $\omega$, $\omega'$, $A$, and $B$ are four complex constants for which $\omega\neq 0$, $\omega'\neq 0$, $\omega/\omega'$ is not real, $f(u+2\omega)=Af(u)$, and $f(u+2\omega')=Bf(u)$. To facilitate investigations of such functions, the following theorem is proved. Let $\alpha=\beta+i\gamma$ and $h=a+ib$ where $\gamma>0$ and $|a|<2\gamma$. Let $R(t)$ be a rational function of $t$ for which $i$ and $-i$ are not poles, and let

$$V_n(z)= \operatorname{cosec}\,(z+n\alpha)R(\cot\,(z+n\alpha)).$$

Then the series in

$$F(z)=\sum_{n=-\infty}^{\infty} e^{hn}V_n(z)$$

converges for all values of $z$ different from the poles of the functions $V_n(z)$, and $F(z)$ is quasi-elliptic with quasi-periods $\pi$ and $\alpha$. Moreover the poles of $F(z)$ are included in the set of poles of the functions $V_n(z)$, and the principal part of $F(z)$ in the neighborhood of a pole $z_0$ is the sum of the principal parts in the neighborhoods of $z_0$ of those of the terms $e^{hn}V_n(z)$ which have poles at $z_0$. Some remarks and an example are given. *R. P. Agnew.*

**Epstein, David I. On the functions of the parabolic cylinder.** Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **BR-19** (1956), i+24 pp.

Separation of the variables of the reduced wave equation

$$(*)\qquad u_{\xi\xi}+u_{\eta\eta}+k^2(\xi^2+\eta^2)u=0$$

leads to two ordinary differential equations which are satisfied by the functions $D_\nu(z)$ of the parabolic cylinder [Erdélyi et al., Higher transcendental functions, vol. II, McGraw-Hill, New York, 1953, p. 116; MR **15**, 419]. In this paper formulas are given expressing regular and single-valued solutions $\omega_\nu(\xi',\eta')=D_\nu(\gamma\xi')D_{-\nu-1}(\gamma\eta')+D_\nu(-\gamma\xi')D_{-\nu-1}(-\gamma\eta')$ of (*) in terms of $\omega$'s with argument $(\xi,\eta)$ in (i) the case where $\xi'$, $\eta'$ are obtained from $\xi$, $\eta$ by a translation in the $xy$-plane, and (ii) the case where the new coordinates are obtained by a rotation about the origin in this plane; here $x=\frac{1}{2}(\xi^2-\eta^2)$, $y=\xi\eta$. In both cases the set of values of $\nu$ for which the addition theorems are valid is the union of two disjoint subsets such that the addition theorem for the functions of one set involves functions of that set only. The results are applied to the asymptotic evaluation of certain integrals appearing in the diffraction problem for radiation of short wavelength from a source in the interior of a parabolic cylinder.

*R. N. Goss* (San Diego, Calif.).

**Liboff, Richard L. A useful integral formula for the initial reduction of the transport equation.** Quart. Appl. Math. **14** (1956), 200–201.

L'auteur utilise un résultat connu sur les harmoniques sphériques pour réduire l'équation intégro-différentielle du transport à un système d'équations différentielles.

*R. Campbell* (Caen).

**See also:** Pyateckiĭ-Šapiro, p. 19; Doetsch, p. 35; Conforto, p. 68; Pao, p. 87; Jaeger, p. 94.

## *Sequences, Series, Summability*

Have

★ **Knopp, Konrad. Infinite sequences and series.** Translated by Frederick Bagemihl. Dover Publications, Inc., New York, 1956. v+186 pp. Paperbound: \$1.75; clothbound: \$3.50.

Der Zweck dieses kleinen Buches ist es, den Leser von den Grundlagen der reellen und komplexen Zahlen ausgehend in die wichtigsten Teile der klassischen Reihenlehre einzuführen. Besonderer Wert wird auf eine sorgfältige Behandlung der verwendeten Grundbegriffe gelegt, sodaß es einem weiter interessierten Studenten leicht fallen wird, selbst noch tiefer in Sondergebiete der Theorie einzudringen. — Das Buch ist meisterhaft klar geschrieben. Druck und Ausführung sind übersichtlich, sodaß insgesamt dem Werk ein großer Leserkreis zu wünschen ist. Eine Behandlung fortgeschrittener, modernerer Themen der Reihenlehre (die für einen Fortsetzungsband angedeutet wird) wäre sehr erfreulich.

In Kapitel 1 werden die wichtigsten Begriffe und Regeln für reelle und komplexe Zahlen und Zahlenmengen wiederholt. Prinzipiell wichtige Dinge werden ausführlich dargelegt, Einzelheiten nur gestreift. Kapitel 2 bringt den Begriff der Folge und speziell der Nullfolge, anschließend allgemeine konvergente und divergente Folgen und die beiden Hauptkriterien für die Konvergenz von Folgen. Ableitung des Begriffs der unendlichen Reihe aus dem der Folge. Kapitel 3 beschäftigt sich mit der Konvergenztheorie und Operationen mit unendlichen Reihen. Nach den Standardkriterien für Reihen mit positiven und mit beliebigen Gliedern wird der Toeplitzsche Satz (hinreichender Teil) für zeileninfinite Matrizen behandelt. Es folgen die üblichen Regeln für das Rechnen mit Reihen, sowie eine kurze Behandlung unendlicher Produkte. In Kapitel 4 werden die gebräuchlichen Sätze über Potenzreihen und die Entwicklung von Funktionen in Potenzreihen gebracht, während in Kapitel 5 die Konvergenztheorie weiter entwickelt wird. Nach den Sätzen von Abel-Dini-Pringsheim werden hier die Kriterien von Abel-Dedekind-Dirichlet für Reihen der Form $\sum a_n p_n$ behandelt und sodann einige einfache Matrixtransformationen angegeben. Hervorzuheben ist eine neue Behandlung [nach Jehle, Math. Z. **52** (1949), 60–61; MR **11**, 241] des Weierstraßschen Kriteriums für Reihen $\sum a_n$, für die

$$\frac{a_{n+1}}{a_n}=1-\frac{\alpha}{n}+c_n \quad \text{mit } \sum |c_n|<\infty$$

gilt. In Kapitel 6 wird die Entwicklung elementarer Funktionen besprochen, und Kapitel 7 beschäftigt sich schließlich mit den Möglichkeiten zur numerischen Auswertung einer Reihe.

Viele Beispiele erläutern die Begriffe und Ergebnisse eines jeden Kapitels. Ein Literatur- und ein Sachverzeichnis beschließen das Buch. *D. Gaier* (Stuttgart).

**Koutský, Zdeněk. Some uses of the number sup $|a_n|^{1/n}$.** Časopis Pěst. Mat. **79** (1954), 273–277. (Czech)

The author considers a differential equation $d\eta/d\xi=f(\xi,\eta)$ in the complex domain with initial condition $\xi=\eta=0$. Applying the calcul des limites he derives, under the assumption of the knowledge of a certain type of majorant for $f(\xi,\eta)$ the best possible estimate for the radius of convergence of the power series of the solution. He further considers the inversion of a mapping

$$w_1=a_{10}z_1+a_{01}z_2+\sum_{i+k>1} a_{ik}z_1^i z_2^k$$

$$w_2=b_{10}z_1+b_{01}z_2+\sum b_{ik}z_1^i z_2^k$$

assuming the knowledge of certain majorants for these power series and certain inequalities for the coefficients of the first order terms which guarantee a non-vanishing Jacobian at $z_1=z_2=0$ and gives again the best estimates for the convergence domain of the power series representing the inverse mapping. *C. Loewner.*

**Tandori, Károly. Über orthogonale Reihen.** Acta Sci. Math. Szeged **16** (1955), 74–76.

Let $a_k(x)$ be an orthonormal system in the interval $[a, b]$ and $\sum c_k a_k(x)$, $\sum c_k^2<\infty$, an orthogonal series. The author proves the following result: if the series is $(C, 1)$

summable almost everywhere, then

$$\lim_{n\to\infty} \sigma_n{}^{\alpha}[(s_\nu - f), x] = 0 \ (0<\alpha<1),$$

almost everywhere, where the $s_\nu$ are the partial sums of the series and the $\sigma_k{}^{\alpha}[(s_\nu-f), x]$ denote the $(C, \alpha)$ means of the sequence $s_\nu(x)-f(x)$. *A. P. Calderón.*

**Karamata, J. Remarque relative à la sommation des séries de Fourier par le procédé de Nörlund.** Publ. Sci. Univ. Alger. Sér. A. 1 (1954), 7–13 (1955).

Hille and Tamarkin [Hille, Bull. Amer. Math. Soc. 38 (1932), 505–528; Hille and Tamarkin, Trans. Amer. Math. Soc. 34 (1942), 157–783] have given a necessary and sufficient condition in order that Nörlund's method should sum a Fourier series to the value of the function at every point of continuity. The author shows that if the condition is satisfied and

$$\limsup_{n\to\infty} \frac{1}{P_n} \sum_1^n \frac{P_\nu}{1+\nu} = \mu,\ P_n = \sum_1^n p_\nu$$

then for $0<\theta<\mu^{-1}$ summability $(C, \theta)$ implies summability $N(p_\nu)$. *A. P. Calderón* (Cambridge, Mass.).

**Cowling, V. F. On Borel summability.** J. London Math. Soc. 31 (1956), 369–373.

The author proves a theorem connecting conditions of analyticity and boundedness of the function $a(w)$ with the region of absolute Borel integral summability of the series $\sum a_n z^n$. *H. G. Eggleston* (Cambridge, England).

**Stipanić, Ernest. Un teorema sulle serie convergenti a termini di segno alternato.** Boll. Un. Mat. Ital. (3) 11 (1956), 242–247.

Some conclusions are drawn from the hypothesis that $u_n>0$ and $u_1-u_2+u_3-u_4+\cdots$ is a convergent series having remainders $r_n$ such that $1<(-1)^{n-1}u_n/r_n<2$. *R. P. Agnew* (Ithaca, N.Y.).

★**Richert, Hans-Egon. Summierbarkeit Dirichletscher Reihen und asymptotische Zahlentheorie.** Colloque sur la Théorie des Nombres, Bruxelles, 1955, pp. 85–92. Georges Thone, Liège; Masson and Cie, Paris, 1956.

Consider the Riesz mean, $C_k(x)=\sum_{1\le n\le x} c_n \log^k x/n$, associated with the infinite series, $\sum_{n=1}^\infty c_n$. The author defines a nonlinear summability method by the condition

$$\int_1^x |C_x(u)-c\log^k u|^2 \frac{du}{u} = o(\log^{2k+1} x).$$

If this condition holds, the series $\sum_{n=1}^\infty c_n$ is said to be $|R, k|^2$-summable to $c$.

Using the Parseval theorem for Fourier integrals, a precise relation is obtained between the $|R, k|^2$-abscissa of convergence of a Dirichlet series, $f(s)=\sum_{n=1}^\infty a_n n^{-s}$, and the Carlson function obtained from the consideration of the mean value $T^{-1}\int_{-T}^{T} |f(s+it)|^2 dt$. *R. Bellman.*

**Jurkat, Wolfgang B. Ein funktionentheoretischer Beweis für O-Taubersätze bei Potenzreihen.** Arch. Math. 7 (1956), 122–125.

Delange [Bull. Sci. Math. (2) 76 (1952), 179–189; MR 14, 634] bewies den $O$-Umkehrsatz der Laplacetransformation, indem er mittels komplexer Integration und eines Satzes von Montel (über die Stetigkeit beschränkter analytischer Funktionen in Winkelräumen) auf einfachere Limitierungsverfahren zurückschloß. Verfasser geht nun im Spezialfall des Abelverfahrens ebenso vor. Er erwähnt jedoch Delange nur am Rande. Auch setzt er sich nicht mit den zahlreichen älteren Arbeiten auseinander [z.B. Hardy und Littlewood, Proc. London Math. Soc. (2) 18 (1919), 205–235; und Bosanquet und Cartwright, Math. Z. 37 (1933), 416–423], in denen ebenfalls mit Hilfe des Montelschen Satzes teilweise sehr allgemeine Umkehrsätze für das Abelverfahren gewonnen wurden.
*K. Zeller* (Tübingen).

**Jakimovski, Amnon. Some Tauberian properties of Hölder transformations.** Proc. Amer. Math. Soc. 7 (1956), 354–363.

Die Folge $\{s_n\}$ heiße $J\left(\alpha;\begin{matrix}\beta,\gamma\\ a,b\end{matrix}\right)$-limitierbar zum Wert $s$, wenn die durch
$t_0=h_0{}^{(\alpha)}$,

$$t_m = h_0{}^{(\alpha)} + \sum_{n=1}^m \frac{h_n{}^{(\alpha)} - [ah_n{}^{(\beta)} + bh_n{}^{(\gamma)}]}{[a(\beta-\alpha)+b(\gamma-\alpha)]n} \ (m=1,2,\cdots)$$

$[\{h_n{}^{(\alpha)}\}$ die zu $\{s_n\}$ gehörige Hölder-Transformation der Ordnung $\alpha]$ erklärte Folge $\{t_m\}$ gegen $s$ strebt für $m\to\infty$. Dabei sind die reellen Parameter $\alpha$, $\beta$, $\gamma$, $a$, $b$ den Bedingungen $a+b=1$, $a\geqq 0$, $\alpha<\beta<\gamma$, $a\alpha+b\beta>0$ unterworfen und $a=1$, $b=0$ liefert ein vom Verf. früher [J. Analyse Math. 3 (1954), 346–381; MR 16, 28] untersuchtes Verfahren $J(\alpha, \beta)$. Die Verfahren $J\left(\alpha;\begin{matrix}\beta,\gamma\\ a,b\end{matrix}\right)$ erweisen sich als zu den Hölder-Verfahren $(H, \alpha+1)$ äquivalente Hausdorff-Verfahren, die sich noch auf Verfahren

$$J\left(\alpha;\begin{matrix}\alpha_1,\cdots,\alpha_n\\ a_1,\cdots,a_n\end{matrix}\right)$$

verallgemeinern lassen, die von $2n+1$ reellen Parametern abhängen. Ein Satz Tauberscher Art sei erwähnt: Notwendig und hinreichend für die $(H, \alpha)$-Limitierbarkeit einer durch irgend ein Hölder-Verfahren limitierbaren Folge $\{s_n\}$ ist die Existenz von $2n$ reellen Zahlen $a_1, \cdots, a_n$; $\alpha_1, \cdots, \alpha_n$ mit $\alpha<\alpha_1<\cdots<\alpha_n$ und $a_1+\cdots+a_n=1$ derart, daß gilt:

$$\lim_{m\to\infty} \{h_m{}^{(\alpha)} - [a_1 h_m{}^{(\alpha_1)} + \cdots + a_n h_m{}^{(\alpha_n)}]\} = 0.$$

*D. Gaier* (Stuttgart).

**Petersen, G. M. Summability methods and bounded sequences.** J. London Math. Soc. 31 (1956), 324–326.

The following theorem is due to A. Brudno [Mat. Sb. N.S. 16 (58) (1945), 191–247; MR 7, 12]: Let every bounded sequence summable by a Toeplitz matrix $A$ also be summable by a Toeplitz matrix $B$. Then it is summable to the same value by $B$ as by $A$. Brudno's proof is complicated and somewhat lengthy; in the present paper the author gives a brief and simple proof by showing that if two regular (Toeplitz) matrices $A$ and $B$ sum a bounded sequence $\{s_n\}$ to different limits, then there exists a bounded sequence which is summed by $A$, but not by $B$. There is also in existence another simplified proof of the theorem due to P. Erdös and G. Piranian which they have not published. *R. G. Cooke* (London).

**Allen, H. S. Transformations of sequence spaces.** J. London Math. Soc. 31 (1956), 374–376.

Let $\mathfrak{L}(\alpha, \beta)$ denote the set of infinite matrices which map a sequence space $\alpha$ into a sequence space $\beta$, and let $\alpha\to\beta$ denote the set of all matrices $A=(a_{n,k})\in\mathfrak{L}(\alpha,\beta)$ such that the series $\sum_{k=1}^\infty a_{n,k}x_k$ $(n=1, 2, \cdots)$ are absolutely convergent for every $\{x_k\}\in\alpha$.

The following notations are used: $C$ is the space of stationary sequences ($x_{k+1}=x_k$ for $k>k_0$), $D$ is the space of alternating sequences ($x_{k+1}=-x_k$ for $k>k_0$), $\sigma_\infty$ is the space of all bounded (complex) sequences, $Z$ is the space of all null sequences ($x_k\to 0$ as $k\to\infty$), $\Gamma$ is the space of all convergent sequences, $\phi$ is the space of all finite sequences ($x_k=0$ for $k>k_0$), and $\sigma_1$ is the space of all sequences $\{x_k\}$ such that $\sum_{k=1}^\infty |x_k|$ converges. The Köthe-Toeplitz dual space of a sequence space $\alpha$ is denoted by $\alpha^*$, and $\alpha$ is perfect if $\alpha^{**}=\alpha$.

The following results are established. Theorem 1. (a) $\mathfrak{L}(C, \beta)$, where $\beta$ is an arbitrary sequence space, is the set of all matrices $A=(a_{n,k})$ such that (i) column vectors of $A$ are in $\beta$, (ii) the series $\sum_i a_{n,i}$ $(n=1, 2, \cdots)$ are convergent, (iii) the sequence $\{\sum_i a_{n,i}\}$ is in $\beta$. (b) $C\to\beta$ is the set of all $A \in \mathfrak{L}(C, \beta)$ with row vectors in $\sigma_1$. Theorem 2. (a) $\mathfrak{L}(D, \beta)$ where $\beta$ is an arbitrary sequence space, is the set of all matrices $A=(a_{n,k})$ such that (i) column vectors of $A$ are in $\beta$, (ii) the series $\sum_i (-1)^i a_{n,i}$ $(n=1, 2, \cdots)$ are convergent, (iii) the sequence $\{\sum_i (-1)^i a_{n,i}\}$ is in $\beta$. (b) $D\to\beta$ is the set of all $A \in \mathfrak{L}(D, B)$ with row vectors in $\sigma_1$. Theorem 3. If (i) $\alpha$ contains $\phi$ and is normal, (ii) $\beta$ is perfect, (iii) $\alpha^{**}\geq\lambda\geq\alpha$, then (a) $\alpha\to\beta=\lambda\to\beta=\mathfrak{L}(\lambda, \beta)=\alpha^{**}\to\beta$ (b) $(\alpha\to\beta)'=\beta^*\to\alpha^*$, the prime denoting the transposed matrix. Corollary 1. The conditions of Theorem 3 are satisfied if we take $\alpha=Z$ and $\lambda=\Gamma$, since $Z^{**}=\sigma_\infty$. Hence, if $\beta$ is perfect, we have

$$Z\to\beta=\Gamma\to\beta=\mathfrak{L}(\Gamma, \beta)=\sigma_\infty\to\beta.$$

Corollary 2. If $\beta$ is perfect, then $(\Gamma\to\beta)'=\beta^*\to\sigma_1$.

These results were suggested to the author on reading some particular cases of Theorems 1 and 2 and Cor. 1 to Theorem 3 in work, at the time unpublished, by H. H. Abu El Makarem. Some of this work has since been published [Review of Economics, Political, Business Studies, Faculty of Commerce, Cairo University 4 (1956), no. 1]. His theorems (II), p. 21, and (VI), p. 33 are particular cases of Theorem 1 of the present paper, and (V), p. 31 is a particular case of Theorem 3, Cor. 1.
*R. G. Cooke* (London).

**See also:** Doetsch, p. 35; Basov, p. 40; Rapoport, p. 40; Horošilov, p. 40.

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## *Approximation, Orthogonal Functions*

**Krylov, V. I. Convergence of algebraic interpolation with respect to roots of Čebyšev's polynomial for absolutely continuous functions and functions of bounded variation.** Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 362–365. (Russian)

Let $f(x)$ be defined in $(-1, 1)$ and let $L_n(x)$ denote Lagrange's interpolating polynomial formed for $f(x)$ at the $n$ zeros of the Čebyšev polynomial of degree $n$. The author establishes the following two theorems. Theorem 1. If $f(x)$ is absolutely continuous in $(-1, 1)$, then for $n\to\infty$ $L_n(x)$ converges to $f(x)$ uniformly with respect to $x$ in $(-1, 1)$. Theorem 2. If $f(x)$ has bounded variation in $(-1, 1)$, then for $n\to\infty$ $L_n(x)$ converges to $f(x)$ at all points of continuity of $f(x)$. *W. E. Milne* (Corvallis, Ore.).

**de Greiff Bravo, Luis. Interpolation and finite differences.** Rev. Acad. Colombiana Ci. Exact. Fis. Nat. **9** (1956), 267–273. (Spanish)

The theory of finite differences is applied to the numerical calculation of polynomials. *E. Frank.*

**Fejér, Leopold. Verschiedene Bemerkungen elementarer Natur über die Grundpolynome, die bei den parabolischen Interpolationen auftreten.** Acta Math. Acad. Sci. Hungar. **6** (1955), 227–240. (Russian summary)

Soient: $a\leq x_1<\cdots<x_n\leq b$, $n$ noeuds d'interpolation dans le segment $[a, b]$, la méthode d'interpolation d'Hermite conduit à considérer les polynomes $h_k(n)$, de degré $(2n-1)$ on $(2n-2)$, dont la dérivée s'annule aux noeuds, le polynome $h_k$ lui-même s'annulant également aux noeuds, sauf en $x_k$ où il prend la valeur un; l'auteur démontre que ce polynome présente en $x_k$ un maximum relatif, en calculant le début du développement de Taylor de $h_k$ au voisinage de $x_k$. On est aussi amené à considérer les polynomes $\mathfrak{H}_k(x)$, de degré $(2n-1)$, possèdant une racine simple en $x_k$, la dérivée étant égale à un, et une racine double aux autres noeuds; alors la somme $\sum \mathfrak{H}_k(n)$ admet une racine dans tout intervalle $(x_k, x_{k+1})$, d'où il suit que toutes ses racines sont réelles. Lorsque, de plus, la répartition des noeuds est normale, et qu'on prend: $a=-1$, $b=1$, on a de plus: $x-1\leq\sum \mathfrak{H}_k(n)\leq x+1$.
*J. Favard* (Paris).

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★ **Kantorowitsch, L. W.; und Krylow, W. I. Näherungsmethoden der höheren Analysis.** VEB deutscher Verlag der Wissenschaften, Berlin, 1956. xi+611 pp. DM 47.00.

Translation of: Približennye metody vysšego analiza. [Gosudarstv. Izdat.-Tehn.-Teor. Lit., Moscow-Leningrad, 1950; MR **13**, 77].

★ **Walsh, J. L. Best-approximation polynomials of given degree.** Proceedings of Symposia in Applied Mathematics. Vol. VI. Numerical analysis, pp. 213–218. Published by McGraw-Hill Book Company, Inc., New York, 1956 for the American Mathematical Society, Providence, R. I. $9.75.

The author gives an introductory survey, with historical references, of the topic of Tchebycheff polynomials corresponding to a finite point set, oriented towards the work of T. S. Motzkin and the author [Trans. Amer. Math. Soc. **78** (1955), 67–81; MR **16**, 585].
*F. V. Atkinson* (Canberra).

**Ėzrohi, I. A. General forms of the remainder terms of linear formulas in multidimensional approximate analysis. I.** Mat. Sb. N.S. **38(80)** (1956), 389–416. (Russian)

The author continues the work of Rémès [Akad. Ukrain. RSR. Inst. Mat. Zb. Prac' **3** (1940), 21–62; **4** (1940), 47–82; MR **2**, 195], Sard [Duke Math. J. **15** (1948); 333–345; MR **10**, 197], and T. G. Ėzrohi [Dopovidi Akad. Ukrain. RSR 1952, 174–179; MR **15**, 511] and obtains expressions, in terms of integrals of partial derivatives, of certain functionals $V(f)$ of functions of several variables. He uses these to obtain exact remainder formulas for interpolation formulas, mechanical quadrature formulas, etc. For example, a continuous linear functional $V(f)$, defined for all functions $f(x, y)$ in $0\leq x, y\leq 1$ with continuous derivatives $\partial^i f/\partial^i x$, $\partial^j f/\partial y^j$ $(i\leq s_1, j\leq s_2)$ which vanishes for all $f$ which are polynomials of degree less than $s_1$ in $x$ and of degree less than $s_2$ in $y$, may be written by means of Stieltjes integrals as

$$\iint \frac{\partial^{s_1} f}{\partial x^{s_1}}\, dF(x, y)+\iint \frac{\partial^{s_2} f}{\partial y^{s_2}}\, dG(x, y),$$

where the kernels $F$, $G$ depend only on $V$. If $V(f)$ is the difference between $f$ and its interpolation polynomial,

$F$, $G$ may be computed. Exact formulations of theorems are too long to be reproduced here. *G. G. Lorentz.*

★ **Sard, Arthur. Function spaces and approximation.** Proceedings of Symposia in Applied Mathematics. Vol. VI. Numerical analysis, pp. 177–185. Published by McGraw-Hill Book Company, Inc., New York, 1956 for the American Mathematical Society, Providence, R. I. $9.75.

The author continues his former work concerning explicit expressions for certain functionals of functions of several variables in terms of integrals of their derivatives. Examples and applications are given. The spaces of functions are those considered by the author in Acta Math. 84 (1951), 319–346; Proc. Amer. Math. Soc. (1952), 732–741 [MR 12, 680; 14, 360]. They are different from those of the preceding review. *G. G. Lorentz.*

**Tournarie, Max. Considérations sur l'histographie des fonctions, notamment dans l'inversion du produit de composition.** C. R. Acad. Sci. Paris 242 (1956), 2509–2512.

The object is to represent (approximately) a given function as the convolution of an interpolation function with a weighted sum of equally spaced point masses. Procedures for choosing the weights, the interpolation function, and the spacing are worked out, as well as some formal properties of the representation. *H. D. Block.*

**Hirschman, I. I., Jr. A note on orthogonal systems.** Pacific J. Math. 6 (1956), 47–56.

The author proves four results generalizing inequalities of Paley [Studia Math. 3 (1931), 226–238; see, e.g., Zygmund, Trigonometrical series, Warszawa-Lwów, 1935, pp. 202–203, 208]. Representative of the type of result proved is the following: If $\omega_1(x)$, $\omega_2(x)$, $\cdots$, is a bounded orthonormal family of functions on the interval $0\leq x\leq 1$ and if $\sum_1^\infty |a_n|<\infty$, then

$$\int_0^1 x^{-\alpha q}\left|\sum_1^\infty a_n\omega_n(x)\right|^q dx\leq A(q,\alpha)\sum_1^\infty |a_n|^q n^{q-2+\alpha q}$$

for $2\leq q<\infty$ and $0\leq\alpha<1/q$. The proof for the case $q=2$ is similar to that in Zygmund for the corresponding Paley result; the general case requires the case $q=2$, Paley's results, and the Riesz interpolation theorem. The paper closes with a proof of a variant of the Riesz-Thorin convexity theorem. *A. E. Livingston.*

**Suetin, P. K. On polynomials orthogonal with a differentiable weight.** Dokl. Akad. Nauk SSSR (N.S.) 106 (1956), 788–791. (Russian)

Let $P_1(z)$, $P_2(z)$, $\cdots$ be the orthonormal system of polynomials with respect to the positive weight function $\varrho(z)$ for the boundary $\Gamma$ of a bounded and simply connected region $G$. If $\varrho^{(p)}\in\text{Lip}\,\alpha$, $\alpha<1$, and if $\Gamma$ is an analytic curve, the author proves that corresponding to each closed subset $F$ of $G$ is a constant $C=C(F)$ such that $|P_n(z)|\leq C/n^{p+\alpha}$ for $z\in F$ and $n=1, 2, 3, \cdots$. If $w=\Phi(z)$ is the analytic function mapping the complement of $G\cup\Gamma$ onto $|w|>1$ and $\Psi$ the function inverse to $\Phi$, let

$$D(w)=\exp\left\{-\frac{1}{4\pi}\int_0^{2\pi}\frac{e^{i\theta}+w}{e^{i\theta}-w}\log\varrho[\Psi(e^{i\theta})]d\theta\right\};$$

then the polynomials $P_n^*(z)$ defined by

$$\sqrt{(\Phi'(w))}/\{D(w)[\Psi(w)-z]\}=\sum_0^\infty P_n^*(z)/w^{n+1}$$

for $z\in G$ and $|w|>1$ satisfy $|P_n^*(z)|\leq C_1(F)/n^{p+\alpha}$ for $z\in F\subset G$. Writing $P_n(z)=P_n^*(z)+I_n(z)$, the author shows that $|I_n(z)|\leq C_2(F)/n^{p+\alpha}$ by an argument like that in Ch. II of P. P. Korovkin [Mat. Sb. N.S. 9(51) (1941), 469–485; MR 3, 114]. *A. E. Livingston* (Seattle, Wash.).

**See also:** Barna, p. 5; Protter, p. 26; Tandori, p. 30; Doetsch, p. 35; Epstein, p. 30; Filin, p. 43; Borovskiĭ, p. 73; Mertens, p. 102; Wunderlich, p. 80; Barenblatt, p. 91.

## *Trigonometric Series and Integrals*

**Cazenave, René. Convergence et sommation d'une série de Fourier correspondant à une fonction analytique.** Ann. Télécommun. 10 (1955), 102–108.

The author discusses the well-known relationship between the trigonometrical series

$$(1)\qquad a_0+\sum_1^\infty a_n\cos nx+b_n\sin nx$$

and the power series

$$(2)\qquad a_0+\sum_1^\infty \alpha_n z^n,\ \alpha_n=a_n-ib_n,$$

and from these considerations he concludes that if the radius of convergence of (2) is larger than 1 then (1) converges absolutely, and that if (1) converges at all points then the same holds for (2) (which is false). After deriving the formulas of Euler-Poisson and Parseval and discussing a vectorial diagram he concludes the paper by apparently asserting that the convergence of (2) for all $|z|=1$ implies $\sum_1^\infty |\alpha_n|<\infty$ {which is false}.

*A. P. Calderón* (Cambridge, Mass.).

**Kinukawa, Masakiti. Some strong summability of Fourier series. II.** Proc. Japan Acad. 32 (1956), 377–382.

[For part I see same Proc. 32 (1956), 86–89; MR 17, 1079.] Let $f(z)=\sum_{n=0}^\infty c_nz^n$ be analytic for $|z|<1$, with boundary function $f(e^{i\theta})$. Let $s_n(\theta)=\sum_{j=0}^n c_je^{ij\theta}$, $t_n(\theta)=nc_ne^{in\theta}$, and let $\sigma_n^\delta(\theta)$ and $\tau_n^\delta(\theta)$ be the $n$th Cesàro mean of order $\delta>-1$ of $\{s_n(\theta)\}$ and $\{t_n(\theta)\}$ respectively. The author considers some consequences of the hypothesis

$$\left(\int_{-\pi}^{\pi}|f'(re^{i\theta})|^p d\theta\right)^{1/p}=O\{(1-r)^{-1+\alpha}(\log 1/1-r)^{-\beta}\}$$

as $r\to 1$ for $2\geq p\geq(1/\alpha)>1$. These are that

$$\sum_{n=1}^\infty|\sigma_n^\delta(\theta)-f(e^{i\theta})|^{1/\alpha}$$

converges almost everywhere if either $\beta>\alpha$ and $\delta>(1/p)-1$ or $\beta>\alpha+(1/p)$ and $\delta=(1/p)-1$ and that $\sum_{n=1}^\infty|\tau_n^\delta|(\theta)|^{1/\alpha}$ converges almost everywhere if either $\beta>\alpha$ and $\delta>1/p$ or $\beta>\alpha+(1/p)$ and $\delta=1/p$. *P. Civin.*

**Jurkat, Wolfgang. Gliedweise Integration und Einzigkeitssätze bei trigonometrischen Reihen.** Math. Ann. 131 (1956), 141–150.

Abel-summability of termwise integrated trigonometrical series in the interval $(0, 2\pi)$ has been discussed by many writers [Zygmund, Trigonometrical series, Warszawa-Lwów, 1935; Chap. IX] and applied to the theory of integrability and unicity. The author treats these problems in a subinterval of $(0, 2\pi)$. For this purpose two

new concepts are introduced. A trigonometrical series

$$(1)\qquad \sum_{n=1}^{\infty} \Re a_n e^{inx}$$

is said to be termwise integrable in an open interval $I$ if it is Abel-summable to a function $f(x)$ almost everywhere in $I$ which is integrable in every subinterval of $I$ and if the termwise integrated series

$$(2)\qquad \sum_{n=1}^{\infty} \Re a_n e^{inx}/(in)^q$$

is Abel-summable to $F^q(x)$ ($q=1, 2, \cdots$) which is the $q$th integral of $f(x)$.

The series (1) is said to be the local Fourier series of a function $f(x)$ in an open interval $I$ if, in every closed subinterval of $I$, the difference of (1) from an ordinary Fourier series is Abel-summable uniformly to zero, and further if (1) is Abel-summable to $f(x)$ almost everywhere in the interval I.

The author proves the following theorems: (I) In an open interval $I$, let the trigonometrical series (1) be Abel-summable everywhere to a function which is integrable in every subinterval of $I$. Then (1) is a local Fourier series in $I$ if and only if the twice integrated series $\sum_{n=1}^{\infty} \Re a_n e^{inx}/(in)^2$ is uniformly Abel-summable in every closed subinterval of $I$. The condition is sufficient for termwise integrability of (1) in $I$ and is also necessary if the condition $a_n=o(n^\alpha)$ ($\alpha\geq 0$) is added. (II) Let the series (2) be Abel-summable uniformly in every subinterval of an open interval $I$ for some $q$. Then if (1) is everywhere Abel-summable to a function which is integrable in every subinterval of $I$ and $F_{2p}(x)$ is continuous in $I$ for all $p\geq 1$, the series (1) is termwise integrable in $I$.

These theorems are extended to the case where there is an "exceptional" set. *S. Izumi* (Sapporo).

**Baženov, G. M. Investigation of expansions in a trigonometric series of expressions of the form**

$$(1-2h\cos z+h^2)^{-n/2}$$

**where $n$ is odd.** Byull. Inst. Teoret. Astr. 6 (1955), 8–24. (Russian)

The expansion under consideration is the following:

$$(1-2h\cos z+h^2)^{-n/2}=\tfrac{1}{2}b_n^{(0)}+\sum_{i=0}^{\infty} b_n^{(i)}\cos iz,$$

in which $n$ is odd. The author derives upper and lower bounds for the coefficients $b_n^{(i)}$ and also obtains expressions which serve as approximations to $b_n^{(i)}$ for a certain range of $i$. Several systematic schemes are developed for the numerical calculation of these coefficients, and the methods are illustrated by concrete examples.

Next there are given upper and lower bounds for the maximum value of the remainder term of the expansion, the results being shown for $n=1$, 3, 5, 7, 9, and 11 by means of diagrams in which $h$ varies from 0.2 to 0.8 and $i$ varies from 0 to 100.

Finally the author considers in detail the specific case of developing in such a series the inverse cube of the distance between two planets. *W. E. Milne.*

**Mohanty, R.; and Nanda, M. On the logarithmic mean of the derived conjugate series of a Fourier series.** Proc. Amer. Math. Soc. 7 (1956), 397–400.

Let $\sigma_n(x)$ be the $n$th logarithmic mean of the derived conjugate Fourier series of $f(x)$ and let

$$\varphi_x(t)=f(x+t)+f(x-t)-2f(x),\ h_x(t)=\varphi_x(t)/4\sin\tfrac{1}{2}t-d.$$

If

$$\int_t^{\pi} |hq(u)|u^{-1}du=o(\log 1/t)\ (t\to 0),$$

then

$$\sigma_{2n}(x)-\sigma_n(x)\to d\pi^{-1}\log 2\ (n\to\infty).$$

*S. Izumi* (Sapporo).

**Gyires, B. Eigenwerte verallgemeinerter Toeplitzscher Matrizen.** Publ. Math. Debrecen 4 (1956), 171–179.

Es sei $f(x)=(f_{\alpha,\beta}(x))$ ($\alpha, \beta=1, \cdots, p$) für $-\pi\leq x\leq +\pi$ eine Hermitesche Matrix stetiger Funktionen mit den Eigenwerten $\lambda_1(x), \cdots, \lambda_p(x)$. Ferner sei

$$T_n(f)=\begin{pmatrix} c_0 & c_1 & \cdots & c_n \\ c_{-1} & c_0 & \cdots & c_{n-1} \\ \cdot & \cdot & \cdots & \cdot \\ c_{-n} & \cdot & \cdots & c_0 \end{pmatrix} \text{mit } c_\nu=\left(\frac{1}{2\pi}\int_{-\pi}^{+\pi} f_{\alpha,\beta}(x)e^{-i\nu x}dx\right),$$

$\nu=0, \pm 1, \pm 2, \cdots$ die zu $f$ gehörige $n$te verallgemeinerte Toeplitzsche Matrix der Ordnung $(n+1)p$. Die Eigenwerte von $T_n(f)$ seien $\lambda_{\nu,k}^{(n)}$ ($\nu=0, \cdots, n$; $k=1, \cdots, p$). In Verallgemeinerung eines Ergebnisses von G. Szegö für $p=1$ [Math. Ann. 76 (1915), 490–503; Math. Termész. Ért. 35 (1917), 185–222] wird gezeigt: Ist $m\leq\lambda_k(x)\leq M$ ($k=1, \cdots, p$), so gilt auch $m\leq \lambda_{\nu,k}^{(n)}\leq M$. Für jede im Intervall $m\leq\lambda\leq M$ erklärte und stetige Funktion $F(\lambda)$ gilt

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{\nu=0}^{n}\sum_{k=1}^{p} F(\lambda_{\nu,k}^{(n)})=\frac{1}{2\pi}\int_{-\pi}^{+\pi}\sum_{k=1}^{p} F(\lambda_k(x))dx.$$

Dieser Satz ermöglicht Aussagen über die Verteilung der $\lambda_{\nu,k}^{(n)}$. *A. Peyerimhoff.*

**Králik, D. Untersuchung der Integrale und Derivierten gebrochener Ordnung mit den Methoden der konstruktiven Funktionentheorie.** Acta Math. Acad. Sci. Hungar. 7 (1956), 49–64. (Russian summary)

The author investigates the degree of approximation of fractional integrals and derivatives by $(C, 1)$ partial sums of their Fourier series, and obtains as corollaries some theorems of Hardy and Littlewood about the Lipschitz classes to which fractional integrals and derivatives belong [Math. Z. 27 (1928), 565–606; Zygmund, Trigonometrical series, Warszawa-Lwów, 1935, chap. 9]. The author's method, due to Alexits [same Acta 3 (1952), 29–42; MR 14, 370] depends on some lemmas on series in a Banach space; a typical one is as follows. Let $\{\phi_n\}$ be a sequence of elements of a Banach space, $\sigma_n$ the arithmetic means of the partial sums of $\sum\phi_n$, and $\sigma_n(\alpha)$ the arithmetic means of the partial sums of $\sum n^{-\alpha}\phi_n$, where $0<\alpha<1$. Then if there is an element $S$ such that $\|\sigma_n-S\|=O(n^{-\beta})$, there is an element $S(\alpha)$ such that $\|\sigma_n(\alpha)-S(\alpha)\|=O(n^{-\alpha-\beta})$ provided that $0\leq\beta<1$ and $\alpha+\beta<1$. Now let $f$ be a periodic function in $L$, $\bar{f}$ its conjugate, $f_\alpha$ its $\alpha$th (Weyl) integral, and $f^{(\alpha)}$ its $\alpha$th derivative ($0<\alpha<1$). Let $\sum A_n(x)$ be the Fourier series of $f$. By $\|\cdots\|_p$ we denote the $L^p$ norm; for $p=\infty$, $L^p$ is to mean the space of continuous functions (contrary to the usual convention). The following theorems are proved. (1) Let $g(x)$ be the function whose Fourier series is $\sum n^{-\alpha}A_n(x)$; then $\|\sigma_n(g)-g\|=O(n^{-\alpha})$; if $f$ and $\bar{f}$ are both in $L$, then $\|\sigma_n(f_\alpha)-f_\alpha\|=O(n^{-\alpha})$. (2) $|\sigma_n(f_\alpha)-f_\alpha|=O(n^{-\alpha})$ at almost every point. (3) If $f\in L^p$ ($1<p<\infty$), then $\|\sigma_n(f_\alpha)-f_\alpha\|_p=O(n^{-\alpha})$ and there is an element $\Psi$ of $L^p$ such that $|\sigma_n(f_\alpha; x)-f_\alpha(x)|\leq\Psi(x)n^{-\alpha}$ for all $x$; it follows as a corollary that $f_\alpha\in \mathrm{Lip}(\alpha, p)$. (4) If $f\in \mathrm{Lip}(\beta, p)$, $0<\beta<1$, with $1\leq p\leq\infty$, and if

$0<\alpha<1$ and $\alpha+\beta<1$, then $\|\sigma_n(f_\alpha)-f_\alpha\|_p=O(n^{-\alpha-\beta})$, and consequently $f_\alpha \in \mathrm{Lip}(\alpha+\beta, p)$. (5) If $f \in \mathrm{Lip}(\alpha, p)$, $0<\alpha\leq 1$, $1<p<\infty$, and $0<\gamma<\alpha$, then $f^{(\gamma)}$ exists and

$$\|\sigma_n(f^{(\gamma)})-f^{(\gamma)}\|_p=O(n^{-\alpha+\gamma});$$

hence $f^{(\gamma)} \in \mathrm{Lip}(\alpha-\gamma, p)$. (6) Let $\mathrm{Lip}(\alpha-0, p)$ be the intersection of all $\mathrm{Lip}(\alpha-\varepsilon, p)$, $\varepsilon>0$; then $f \in \mathrm{Lip}(\alpha-0, p)$, $0<\alpha\leq 1$, if and only if to every $\varepsilon>0$ corresponds $\Psi_\varepsilon \in L^p$ such that

$$|\sigma_n(f;\, x)-f(x)|\leq n^{-\alpha+\varepsilon}\Psi_\varepsilon(x),\quad |\tilde{\sigma}_n(\tilde{f};\, x)-\tilde{f}(x)|\leq n^{-\alpha+\varepsilon}\Psi_\varepsilon(x).$$

(7) Let $s_n$ be the ordinary partial sums of the Fourier series of $f$. Then $f \in \mathrm{Lip}(\alpha-0, p)$ if and only if

$$\|s_n-f\|_p=o(n^{-\alpha+\varepsilon})$$

for every positive $\varepsilon$. *R. P. Boas, Jr.* (Evanston, Ill.).

**See also:** Pisot, p. 19; Karamata, p. 31; Richert, p. 31.

## *Integral Transforms*

Have

★ **Doetsch, Gustav. Handbuch der Laplace-Transformation. Bd II. Anwendungen der Laplace-Transformation. 1. Abteilung.** Birkhäuser Verlag, Basel und Stuttgart, 1955. 436 pp. 56.15 francs suisses.

This is the second volume of a three volume treatise on the Laplace transform. The first volume [1950; MR 13, 230] was devoted to the theoretical aspects of the subject. The present volume and volume III (which is promised for the near future) are concerned with applications. As in the first volume, the exposition is carried out in great detail, and the demands upon the reader's mathematical background are kept to a minimum. Moreover, there is a wealth of illustrative examples. Consequently this treatise should prove of use to a very wide audience.

Chapter 1 is devoted to a review of the basic rules for the Laplace transform, particularly those needed in the sequel. The remainder of the book is then divided into three parts. Part I (chapters 2–10) is concerned with the theory of asymptotic expansions. The treatment here, extending over 200 pages, is unusually complete and virtually every well-known method is explored. The applications are too numerous to list but include many results from the theory of special functions, a study of the spectrum of the hydrogen atom, a brief discussion of renewal theory, etc. The author often derives a formula by several different methods, a practice which should be particularly illuminating for those encountering these techniques for the first time. Moreover, if several related forms of a result are needed for applications, then all are given. Thus many theorems are stated in the language of the bilateral Laplace transform and also in the language of the Mellin transform. Part II (chapters 11 and 12) is devoted to convergent expansions and includes a particularly extensive treatment of factorial series. Also treated are a selection of formulas from the theory of special functions. Part III (chapters 12–16) deals with ordinary differential equations, particular attention being paid to engineering applications — for example the theory of servomechanisms and network theory. In this connection the author has taken care to give the American and English (when different) equivalents of the technical terms used, a practice which will greatly facilitate the book's use by English speaking readers. There is a brief appendix devoted to a proof of the Lagrange-Burmann theorem. The historical and bibliographical notes are collected in a section at the end. References to papers and books are in terms of the bibliography given at the end of volume I. *I. I. Hirschman* (St. Louis, Mo.).

★ **Heinz, Erhard. On an inequality for linear operators in a Hilbert space.** Report of an international conference on operator theory and group representations, Arden House, Harriman, N. Y., 1955, pp. 27–29. Publ. 387, National Academy of Sciences-National Research Council, Washington, D. C., 1955.

The object of the paper is to give an especially simple proof of the inequality $|(Qx, y)|\leq\|A^\sigma x\|\cdot\|B^{1-\sigma}x\|$ for $0\leq\sigma\leq 1$, $A$, $B$, positive definite operators, $Q$ an arbitrary operator with adjoint $Q^*$ such that for $x \in \mathfrak{D}_A\subseteq\mathfrak{D}_Q\cap\mathfrak{D}_{Q^*}$, $y \in \mathfrak{D}_B\subseteq\mathfrak{D}_Q\cap\mathfrak{D}_{Q^*}$, $\|Qx\|\leq\|Ax\|$, $\|Q^*y\|\leq\|By\|$. The inequality is due to the author [Math. Ann. **123** (1951), 415–438; MR **13**, 471] and other proofs had been given by Kato [ibid. **125** (1952), 208–212; MR **14**, 766] and Dixmier [ibid. **126** (1953), 75–78; MR **15**, 39]. Putting $Q(s)=B^{-(1-s)}QA^{-s}$, if $A$, $B$ are strictly positive, $f(s)=(y, Q(s)x)$ proves to be an integral function and $|f(0+it)|\leq\|y\|\cdot\|x\|$, $|f(1+it)|\leq\|x\|\cdot\|y\|$ and $f(\sigma+it)$ is bounded for $0\leq\sigma\leq 1$. Hence it is easy to prove $\|f(s)\|\leq\|x\|$, $\|y\|$ in the strip. This proves the theorem in this special case. The general case is treated by straightforward approximation. *František Wolf.*

**Eckart, Gottfried. Statistische Beschreibung der dielektrischen Turbulenz in der Troposphäre.** Abh. Bayer. Akad. Wiss. Math.-Nat. Kl. (N.F.) no. **74** (1955), 34 pp.

A collection of well-known results on multiple Fourier series, Fourier integrals, Mellin, and two-sided Laplace transforms. It is apparently the author's intention to use these results in a forthcoming publication.

*S. Chandrasekhar* (Williams Bay, Wis.).

**Levin, B. Transformations of Fourier and Laplace types by means of solutions of differential equations of second order.** Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 187–190. (Russian)

In this note the author announces some results concerning generalized Fourier and Laplace Transforms. If $y(x, \lambda)$ denotes a solution of

$$(1)\qquad y''-q(x)y+\lambda^2 y=0,$$

where $q(x)$ is a complex-valued function on $0\leq x<\infty$ satisfying the condition $\int_0^\infty |q(x)|(1+x^2)\,dx<\infty$, the author obtains an inversion of the transformation

$$(2)\qquad \psi(\lambda)=(2\pi)^{-\frac{1}{2}}\,\mathrm{l.i.m.}\int_0^\infty f(x)y(x, \lambda)\,d\lambda$$

for $f(x) \in L^2(0, \infty)$ in the form

$$(3)\qquad f(x)=(2\pi)^{-\frac{1}{2}}\,\mathrm{l.i.m.}\int_{-\infty}^\infty \psi(\lambda)z(x, \lambda)\,d\lambda,$$

where $\psi(\lambda) \in L^2(-\infty, \infty)$. The function $z(x, \lambda)$ is obtained as follows. If $G(x, \lambda)=y_1(x, \lambda)-e^{i\lambda x}$ (where $y_1$ is the solution at (1) such that $\lim_{x\to\infty} G(x, \lambda)=0$), then $K(x, t)=(2\pi)^{-1}\,\mathrm{l.i.m.}\int_{-\infty}^\infty G(x, \lambda)e^{i\lambda t}d\lambda$ exists and is zero for $t<x$. Then $z(x, \lambda)=e^{-i\lambda x}+\int_0^x R(s, x)e^{-i\lambda s}ds$, where $R(s, x)$ is the resolvent kernel of the integral equation $X(x)=f(x)+\int_x^0 K(s, x)f(s)ds$. He then rewrites the mutually inverse formulas (2) and (3) in a form generalizing the Laplace rather than the Fourier transform and states some results concerning the relation between (3) and the expansion of $f$ in the eigenfunctions of various boundary value problems for (1) on the interval $0\leq x<\infty$. The expansion (3) is not an eigenfunction expansion in general. *R. R. Kemp.*

**Tricomi, Francesco G. Sull'inversione dell'ordine di integrali "principali" nel senso di Cauchy.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 18 (1955), 3–7.

The author gives a proof of the formula

$$\int_a^b \frac{dz}{z-x}\int_a^b \frac{F(x,y,z)}{y-z}dy-\int_a^b dy\int_a^b \frac{F(x,y,z)}{(y-z)(z-x)}dz=-\pi^2F(x,x,x),$$

where all the integrals involved are Cauchy principal values, under the following conditions. Write

$$\Phi(x,y,z)=\frac{F(x,y,z)-F(x,z,z)}{y-z},$$

$$\Psi(x,z)=\frac{F(x,z,z)-F(x,x,x)}{z-x}.$$

Then $\Phi(x,y,z)$ is required to be summable with respect to $y$ and with respect to $z$ and to be such that

$$\int_a^b \frac{\Phi(x,y,z)-\Phi(x,y,x)}{z-x}dz$$

exists as a Lebesgue integral, and $\Psi(x,z)$ is required to belong to the class $L_p$ for some $p>1$ (misprinted $p<1$ in the paper). *F. Smithies* (Cambridge, England).

**Isaacs, G. L. Comparison theorems for Laplace integrals.** J. London Math. Soc. 31 (1956), 282–300.

Let $s_1$ and $s_2$ be complex numbers and let $\mu$ and $\alpha$ be real numbers for which $\mu\geqq 0$ and $\alpha-\mu\geqq -1$. In terms of a given complex-valued function $A(u)$, other functions $A_\gamma(u)$ are defined by $A_0(u)=A(u)$;

$$A_\gamma(u)=\frac{1}{\Gamma(\gamma)}\int_0^u (u-t)^{\gamma-1}A(t)dt$$

when $\gamma>0$ and $A(t)$ is Lebesgue integrable over each finite interval $0<t<u$; and

$$A_\gamma(u)=\frac{1}{\Gamma(\gamma+1)}\int_0^u (u-t)^\gamma dA(t)$$

when $-1<\gamma<0$ and $A(t)$ has bounded variation over each finite interval $0<t<u$. When $A_{-1}(t)dt$ appears under an integral sign, it stands for $dA(t)$; in such cases $A(t)$ is assumed to have bounded variation over each finite interval $0<t<u$. Let $r\geqq 0$. Let $P_1$ denote the proposition that the Laplace integral

$$\int_0^\infty e^{-us_1}A_\alpha(u)du$$

is evaluable by the Cesàro method $(C,r)$ [or $|C,r|$] of order $r$. Let $P_2$ denote the proposition that the Laplace integral

$$\int_0^\infty e^{-us_2}A_{\alpha-\mu}(u)\,du$$

is evaluable $(C,r+\mu)$ [or $|C,r+\mu|$]. The question whether $P_1$ implies $P_2$ and the question whether $P_2$ implies $P_1$ are completely answered. The answers depend upon $s_1$, $s_2$, $r$, and $\mu$, and include known results to which references are given. Sixteen cases are considered. For example if $s_1=\sigma_1+it$, and $s_2=\sigma_2+it_2$ where $\sigma_1<0$ and $\sigma_1<\sigma_2<0$, then $P_1$ implies $P_2$ where $\mu$ is an integer but $P_1$ does not imply $P_2$ when $\mu$ is not an integer. *R. P. Agnew.*

**Cambi, Enzo. Inverse Laplace transforms expressed as Neumann series.** J. Math. Phys. 35 (1956), 114–122.

Das Problem, die inverse Laplace transformierte zu gewinnen, verlangt die Auflösung der Integralgleichung

$$F(p)=\int_0^\infty e^{-pt}f(t)\,dt.$$

Beachtet man nun, daß

$$\int_0^\infty e^{-pt}J_n(t)dt=\frac{(\sqrt{(1+p^2)}-p)^n}{\sqrt{(1+p^2)}}(n\geqq 0)$$

unter Verwendung von $p=\sinh u$ sich in der Form

$$\int_0^\infty e^{-t\sinh u}J_n(t)dt=\frac{e^{-nu}}{\cosh u}$$

schreiben läßt, so ist es naheliegend, für die gesuchte Funktion $f(t)$ den Ansatz

$$f(t)=a_0J_0(t)+a_1J_1(t)+a_2J_2(t)+\cdots$$

zu verwenden, da die Gewinnung der Koeffizienten $a_n$ im wesentlichen die Entwicklung der gegebenen Funktion in eine Potentzreihe verlangt. Für eine dadurch nahegelegte numerische Behandlung einer Aufgabe, wirkt sich der Umstand, daß $J_n(t)$ mit wachsendem $n$ rasch abnimmt, sehr günstig aus. Als Beispiel wird die Telegraphengleichung behandelt, wobei sich herausstellt, daß namentlich im Falle, daß die Leitung annähernd verzerrungsfrei ist, die erhaltene Reihe rasch konvergiert. *P. Funk.*

**Selfridge, R. G. Generalized Walsh transforms.** Pacific J. Math. 5 (1955), 451–480.

The Walsh functions have been generalized by Chrestenson [same J. 5 (1955), 17–31; MR 16, 920] as follows. Let $\alpha$ be a fixed integer greater than 1; for $p/\alpha\leqq x-[x]<(p+1)/\alpha$, define $\varphi_0(x)=\exp(2\pi ip/\alpha)$, $\varphi_n(x)=\varphi_0(\alpha^n x)$, $n\geqq 0$. If $n=n_0+n_1\alpha+\cdots+n_\nu\alpha^\nu$, $n_i=0,1,\cdots,\alpha-1$, define $\psi_n(x)=\varphi_0^{n_0}(x)\varphi_1^{n_1}(x)\cdots\varphi_\nu^{n_\nu}(x)$. The author extends the system $\{\psi_n(x)\}$ to $\{\psi_y(x)\}$, $x$, $y\geqq 0$, as in the reviewer's paper for $\alpha=2$ [Trans. Amer. Math. Soc. 69 (1950), 66–77; MR 13, 126] by the definition $\psi_y(x)=\psi_{[y]}(x)\psi_{[x]}(y)$. With $\psi_y(x)$ there is defined a Walsh-Fourier transform for functions in $L_p(0,\infty)$, $1\leqq p\leqq 2$. Some of the classical Fourier transform theorems are carried over, including the Riemann-Lebesgue and Plancherel theorems. For $f\in L_1$, $(C,1)$ summability of the transform is shown to hold at every point which (1) belongs to the Lebesgue set of $f$, and (2) belongs to an interval in which $f$ is essentially bounded. There exist functions for which it fails at some points satisfying (1) but not (2). The question of $(C,1)$ summability almost everywhere is left open. {It appears to the reviewer that this question is easily reducible to the same one for Walsh-Fourier series, contrary to the author's remarks on p. 468.} *N. J. Fine.*

**Fox, Charles. A classification of kernels which possess integral transforms.** Proc. Amer. Math. Soc. 7 (1956), 401–412.

Two functions $f(u)$ and $F(s)$ are said to be Mellin transforms of each other of they are related in the following manner

$$F(s)=\int_0^\infty f(u)u^{s-1}du \text{ and } f(u)=(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty}F(s)u^{-s}ds.$$

The author proves the following theorem and his converse: If on the line $s=\frac{1}{2}+it$ $(-\infty<t<\infty)$, $|G(s)s^{c+1}|$ ($c$ is a real constant $>0$) and $|H(s)|$ are bounded;

$$K(s)=H(s)/(m+ns),$$

where $m$ and $n$ are constants and $K(s)$ has no poles on

the line $s=\frac{1}{2}+it$; $H(s)H(1-s)=1$ and

$$f(x)=\int_0^\infty k(ux)g(u)du$$

($k$ and $g$ are then Mellin transforms of $K(s)$ and $G(s)$), then $f(u)\in L^2$ and

$$g(x)=a\int_0^\infty h(ux)f(u)du+b\frac{d}{dx}\int_0^\infty xk(ux)\frac{d}{du}(uf(u))du,$$

where $a=m^2+mn$ and $b=n^2$.

He also proves a symmetrical inversion formula and an analogue of Parseval's theorem and gives a kernel classification. *W. Saxer* (Zürich).

★ **Rellich, Franz. Linearly perturbed operators.** Report of an international conference on operator theory and group representations, Arden House, Harriman, N. Y., 1955, pp. 30–36. Publ. 387, National Academy of Sciences-National Research Council, Washington, D. C., 1955.

The aim of this paper is to deduce conditions for the concavity of the lowest eigenvalue $\lambda_0$ of $A+\varepsilon B$ as a function of $\varepsilon$. The concavity follows easily for bounded operators, but for unbounded $A$, $B$, it will take place only under certain conditions. A theorem is given listing three alternative conditions guaranteeing the concavity. As an example we quote the following: If $A$ in $\mathfrak{D}$ is essentially selfadjoint and bounded below, and $B$ in $\mathfrak{D}$ symmetric and for some $\alpha$, $\beta$ non-negative and all $\mu\in\mathfrak{D}$, $\|Bu\|^2\leq\alpha\|u\|^2+\beta(u, Au)$, then $A+\varepsilon B$ is essentially selfadjoint and the lowest eigenvalue of its selfadjoint extension is a concave function of $\varepsilon$. Two examples are given: the helium operator and the following eigenvalue problem $-u''+\varepsilon x^{-2}u=\lambda$, $0<x<\infty$, $u_0\cos\alpha+u_1\sin\alpha=0$ at $x_0$. *František Wolf* (Berkeley, Calif.).

**See also:** Byrd, p. 90; Morgan, p. 85.

## *Ordinary Differential Equations*

**Mitrinovitch, Dragoslav S. Compléments au Traité de Kamke. III.** Boll. Un. Mat. Ital. (3) **11** (1956), 168–171.

[Pour les partis I–II voir Jbr. Deutsch. Math. Verein. **58** (1956), Abt. 2, 58–60; Bull. Soc. Math. Phys. Serbie **7** (1955), 161–164; MR **17**, 1086.] Continuant la liste des exemples qu'il projette d'adjoindre à ceux donnés par Kamke dans son traité [Differentialgleichungen, Lösungsmethoden und Lösungen, Bd. I, Akademische Verlagsgesellschaft, Leipzig, 1942; pour une analyse de la 3ième éd. voir MR **9**, 33], l'auteur recherche les cas d'intégrabilité élémentaire de l'équation différentielle:

$$x^\alpha y^\beta\left(\frac{dy}{dx}\right)^\gamma+Ay^\lambda\left(\frac{dy}{dx}\right)^\mu+Bx^\nu=0$$

et, plus généralement, de

$$\sum_{n=1}^N A_n x^{\alpha_n}y^{\beta_n}\left(\frac{dy}{dx}\right)^{\gamma_n}=0.$$

*R. Campbell* (Caen).

**Wintner, Aurel. Sur le calcul des limites de Cauchy dans la théorie des équations différentielles ordinaires.** C. R. Acad. Sci. Paris **242** (1956), 1106–1107.

The author considers $dw/dz=f(z, w)$ with solution $w(z)$ satisfying $w(0)=0$. Let $f(z, w)$ be analytic for $|z|<a$ and $|w|<b$. Let $F(r, s)$ be continuous for $(0\leq r<a,\ 0\leq s<b)$. Suppose that $|f(x, w)|\leq F(|z|, |w|)$. Let $s(r)$ be a solution of $ds/dr=F(r, s)$ with $s(0)=0$. Let $s(r)$ exist for $0\leq r<c$. He proves $w(z)$ is analytic in $|z|<c$. *N. Levinson.*

**Mikeladze, Š. E. On discontinuous solutions of ordinary quasi-differential equations.** Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 641–644. (Russian)

The equations are of the form

$$Ly\equiv\varrho_0\frac{d}{dx}\varrho_1\frac{d}{dx}\cdots\varrho_{n-1}\frac{d}{dx}\varrho_n y+py=0,$$

where the coefficients are continuous functions of $(x, \alpha, \beta, \cdots)$ ($\alpha$, $\beta$, $\cdots$ parameters) in a region including the interval $a\leq x\leq b$. The quasi-derivatives of $y$ are defined thus: $y^{[0]}=p_n y$,

$$y^{[n-k]}=\varrho_k\frac{d}{dx}\varrho_{k+1}\frac{d}{dx}\cdots\varrho_{n-1}\frac{d}{dx}\varrho_n y\ (k=1, \cdots, n-1).$$

First a "fundamental" set of solutions is constructed at $x=s$. Then it is shown how to form a linear combination having, along with its quasi-derivatives, desired jumps at specified points in $[a, b]$. Next a solution is found for the nonhomogeneous equation $Ly=q(x, \alpha, \beta, \cdots)$. In conclusion, certain special cases are noted. *F. A. Ficken.*

**Petropavlovskaya, R. V. On continuation of solutions of a system of differential equations.** Vestnik Leningrad. Univ. **11** (1956), no. 7, 40–59. (Russian)

Let (1): $\dot{x}=f(t, x)$, $x$, $f$ being $n$-vectors; let $G$ be a domain of $x$ space, $J=(p, q)$ an open interval (finite or infinite); the assumption is made that each component of $f$ is continuous in $J\times G$ except at isolated values of $t$ in $J$ (not necessarily the same for all components) where it tends to infinity. A solution is a continuous vector satisfying the equation at all non singular points. Theorem 1: Assume that $f$ satisfies in the neighborhood of each $(t, x)\in J\times G$ a condition of Carathéodory, $\|f\|\leq M(t)$, $M$ summable, and that a solution is defined in the left neighborhood of $t=T\in J$ (in any case $T$ is a finite value); then, when $t\to T-0$, either $x(t)$ tends to an interior point of $G$ or enters a neighborhood of the boundary of $G$ and stays there for all $t$ in a certain left neighborhood of $T$ (if $f$ is continuous in $J\times G$ this reduces to a theorem of Erugin [Prikl. Mat. Meh. **15** (1951), 55–58; MR **12**, 611]; the last part of the statement was overlooked by J. L. Massera in his review of Erugin's paper in Zbl. Math. **42** (1952), 93 and probably also by J. G. Wendel in the MR review}. Theorem 2: Assume that $G$ consists of the whole $x$ space with the exception of a set of isolated points and that there is an $r\geq 0$ such that if $t\in J$, $\|x\|\geq r$, we have $\|f\|\leq A(t, \|x\|)$, where all solutions of $\dot{z}=A(t, z)$ are defined in a left neighborhood of $q$; then any solution of (1) is either continuable in a left neighborhood of $q$ or else there is a $T<q$ such that if $t\to T$, $x(t)$ tends to one of the boundary points of $G$. Theorem 3: If the inequality in Theorem 2 is replaced by $\|f\|\leq C(t)\cdot L(\|x\|)$, where $C$ is continuous except at isolated values of $t$ and summable in any interval completely interior to $J$, and $L$ is continuous $\int_r^\infty dr/L(r)=\infty$; then each solution of (1) either exists in $J$ or there is a $T\in J$ such that for $t\to T$ the solution tends to one of the boundary points of $G$. If $J$ is the whole line, $r=0$, $C=1$ this leads to a criterium of Wintner [Amer. J. of Math. **67** (1945), 277–284; MR **6**, 225]; examples show that: (i) Theorem 2 is essentially more general than Theorem 3; (ii) even if $f$ is continuous everywhere it is in general too restrictive to choose $C$

continuous in Theorem 3 (in particular this shows that Theorem 3 is essentially more general than Wintner's criterium); (iii) no criterium relying only on the norm of $f$ can be a necessary condition, because an example ($n=1$) is given where the solutions of $\dot{x}=f$ are continuable while those of $\dot{x}=-f$ are not. Other similar results are proved of which the following is typical: If there is an $r>0$ such that when $t \in J$, $\|x\|\geqq r$, we have $|x\cdot f|\geqq C(t)\|x\|\cdot L(\|x\|)$ with $C$ continuous and positive $\int_T^q Cdt=\infty$ for any $T \in J$, $L$ continuous and positive $\int_r^\infty dr/L(r)<\infty$; then if $(T_1, T_2)$ is the maximum interval where a certain solution may be continued, either $T_2<q$ or there exists a $t_0$, $T_1\leqq t_0<T_2$ such that $\|x(t)\| >r$ for $T_1<t<t_0$, $<r$ for $t_0<t<T_2$.

*J. L. Massera* (Montevideo).

**Krasnosel'skiĭ, M. A.; and Kreĭn, S. G. On a class of uniqueness theorems for the equation $y'=f(x, y)$.** Uspehi Mat. Nauk (N.S.) 11 (1956), no. 1(67), 209–213. (Russian)

Consider the problem $y'=f(x, y)$, $y(x_0)=y_0$, with $|f(x, y_1)-f(x, y_2)|(x-x_0)\leqq k|y_1-y_2|$ on $R$: $x_0\leqq x\leqq x_0+a$, $|y-y_0|\leqq b$. If $f(x, y)$ is continuous on $R$ then an improvement by Perron [Math. Z. 28 (1928), 216–219] of a theorem of Rosenblatt yields uniqueness of the solution if $k\leqq 1$. The principal result in the paper under review is that if also $|f(x, y_1)-f(x, y_2)|\leqq p|y_1-y_2|^\alpha$ on $R$, with $p$ fixed and $0<\alpha<1$, then uniqueness follows if in the first condition merely $k(1-\alpha)<1$. Using appropriate Banach spaces, the authors obtain similar uniqueness theorems for an integro-differential equation, for a system of ordinary differential equations, and for $y'=f(x, y)$ with $y_0=\lim y(x)$ as $x\to\infty$.

*F. A. Ficken* (Knoxville, Tenn.).

**Picone, Mauro; e Ghizzetti, Aldo. Integrazione dei sistemi degeneri di equazioni differenziali ordinarie lineari a coefficienti costanti.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 19 (1955), 195–199.

The authors investigate the solubility and the number of independent solutions of the system

$$\sum_{k=1}^{p} f_{hk}(D)y_k=\varphi_h(x) \quad (h=1, \cdots, p),$$

where the $f_{hk}(D)$ are polynomials with constant coefficients in $D=d/dx$. The results depend on the rank of the matrix $\|f_{hk}(D)\|$ rather as in the theory of linear algebraic equations. A special case is that in which there are more or fewer equations than unknowns. The method is one of successive reduction to a form in which the unknowns fall into groups, those of the $i$th group not occurring after the $i$th equation.

*F. V. Atkinson* (Canberra).

**Bihari, I. A generalization of a lemma of Bellman and its application to uniqueness problems of differential equations.** Acta Math. Acad. Sci. Hungar. 7 (1956), 81–94. (Russian summary)

The keystone of the paper is the following integral inequality: Let $u(x)$, $v(x)$ be two positive continuous functions in $a\leqq x\leqq b$, and $k\geqq 0$, $m\geqq 0$ be two constants. Further, let $g(u)$ be a non-negative non-decreasing continuous function for $u>0$. Then the inequality $u(x)\leqq k+m\int_a^x v(t)g(u(t))dt$, $a\leqq x\leqq b$, implies the inequality $u(x)\leqq G^{-1}(G(k)+m\int_a^x v(t)dt)$, $a\leqq x\leqq b$, where $G(u)=\int_{u_0}^u dt/g(t)$, $u_0>0$, $u>0$. The case $g(t)=t$ was given by the reviewer [Duke Math. J. 10 (1943), 643–647; MR 5, 145] and applied to stability theory. Subsequently, this case was used by Nemyckiĭ and Stepanov [Qualitative theory of differential equations, OGIZ, Moscow-Leningrad, 1947; MR 10, 612] in the investigation of uniqueness of solutions and dependence of the solution upon a parameter.

In this paper, the generalized inequality is applied to furnish simple unified proofs of the classical uniqueness theorems of Nagumo, Osgood and Perron, and a result in the opposite direction due to Tamarkin-Lavrent'ev.

*R. Bellman* (Santa Monica, Calif.).

**Hirasawa, Yoshikazu. On singular perturbation problems of non-linear systems of differential equations. III.** Comment. Math. Univ. St. Paul. 4 (1955), 93–104.

The author continues earlier investigations [same Comment. 3 (1955), 115–122; 4 (1955), 15–23; MR 17, 366]. He now considers a vector system $\dot{x}=f(x, y, z, t, \varepsilon)$, $\varepsilon\dot{y}=g(x, y, z, t, \varepsilon)$, $\varepsilon\dot{z}=h(x, y, z, t, \varepsilon)$; the degenerate system $\dot{x}=f(x, y, z, t, 0)$, $0=g(x, y, z, t, 0)=h(x, y, z, t, 0)$ is assumed to have a smooth solution $(X(t), Y(t), Z(t))$ on $t_1\leqq t\leqq t_2$. Let $p(\varepsilon)$, $q(\varepsilon)$, $r(\varepsilon)$ be given functions tending to zero with $\varepsilon$ ($>0$), and let to be a given point in $t_1\leqq t\leqq t_2$. The main theorem, proved under suitable conditions on $f$, $g$ and $h$, states that for sufficiently small $\varepsilon$ the system has a solution $(x(t, \varepsilon), y(t, \varepsilon), z(t, \varepsilon))$ on $t_1\leqq t\leqq t_2$, such that $x(t_0, \varepsilon)=X(t_0)+p(\varepsilon)$, $y(t_1, \varepsilon)=Y(t_1)+q(\varepsilon)$, $z(t_2, \varepsilon)=Z(t_2)+r(\varepsilon)$; also all such solutions tend uniformly to

$$(X(t), Y(t), Z(t))$$

as $\varepsilon\to+0$.

*G. E. H. Reuter* (Manchester).

**Sibuya, Yasutaka. Sur les solutions bornées d'un système des équations différentielles ordinaires non linéaires à coefficients périodiques.** J. Fac. Sci. Univ. Tokyo. Sect. I. 7 (1956), 333–341.

Let latin capitals denote $n$-component complex vectors and $\eta=(\eta_\alpha)$ $(\alpha=1, \cdots, \varrho)$ a vector of complex parameters. The system is (1) $dX/dt=F(X; t)$ where for $\|X\|=\max_k |x_k|<\delta$ $F$ is continuous in $(X; t)$ and holomorphic in $x_k$ and $\omega$-periodic in $t$, with $F(0; t)=0$. Let (2) $X=G(t; \eta)$ satisfy (1) with $G$ continuous in $(t, \eta)$ and holomorphic in the $\eta_\alpha$ for $\|\eta\|<\delta'$, and $0\leqq t<\infty$, with $G(t; 0)=0$ and (3) $\|G(t; \eta)\|\leqq M$ where $M$ is independent of $t$ and $\eta$. The principal result is that as $t\to\infty$ $G$ approaches asymptotically an almost-periodic solution of (1). The result applies to the initial value problem with the vector of initial values. Finally, in order that (2) be almost-periodic in $t$ uniformly for $\|\eta\|\leqq\delta'$ it is necessary and sufficient that (3) hold for $\|\eta\|\leqq\delta'$ and $-\infty<t<\infty$. The argument uses an earlier reduction of the system [same J. 7 (1954), 243–254; MR 16, 1026].

*F. A. Ficken.*

**Olver, F. W. J. The asymptotic solution of linear differential equations of the second order in a domain containing one transition point.** Philos. Trans. Roy. Soc. London. Ser. A. 249 (1956), 65–97.

Results of two kinds are obtained. There is first presented a general discussion of the question as to when a differential equation of the type (i):

$$d^2w/dz^2=[up(z)+q(z)]w,$$

wherein $z^2p(z)$ and $z^2q(z)$ are analytic in a closed neighborhood $\mathcal{D}$ of $z=0$ and $u$ is a large positive parameter, has asymptotic solutions of the form

$$P(u, z)[1+\sum_1^\infty A_s(z)u^{-s}]+u^{-1}\partial P(u, z)/\partial z\sum_0^\infty B_s(z)u^{-s},$$

in which $A_s$ and $B_s$ are analytic in $\mathcal{D}$ and $P(u, z)$ is "well-known." Since any equation of type (i) may be simply transformed into the form (ii): $d^2w/dz^2=[uz^n+r(z)]w$,

$n=-2, -1, 0, \cdots$, the condition is imposed that $P(u, z)$ be of the form $\chi(z\psi(u))$. The case $n=-2$ has been investigated previously by the reviewer and McKelvey [Canad. J. Math. 8 (1956), 97–104; MR 17, 736] and also by the author [same Trans. 247 (1954), 307–327; MR 16, 695]. It is shown that if $p(z)\not\equiv 0$, a necessary and sufficient condition that an equation of type (i) have a solution of the form $\chi(z\psi(u))$ is that $p(z)=\alpha z^n$ and $q(z)=\beta z^n+\gamma z^{-2}$, where $n=-1, 0, \cdots$, and $\alpha$, $\beta$, and $\gamma$ are constants. It is next determined for which $r(z)$ the coefficients $A_s(z)$ and $B_s(z)$ in the asymptotic solutions of (ii) are analytic. Apart from several special cases not characterized, the results are these: if $n=\gamma=0$ or $n=1$ and $\gamma=0$, $r(z)$ may be any analytic function; otherwise, $r(z)$ is restricted to be of the form (iii): $\gamma z^{-2}+z^n h(z^{n+2})$, wherein $h(t)$ is analytic. Thus, if complete asymptotic series for solutions of equations of type (i) with turning points of higher order than one are to be found when $q(z)$ is not severely restricted, the search for related equations must be broadened. This has, in fact, been done in the case $n=2$ by McKelvey [Trans. Amer. Math. Soc. 79 (1955), 103–123; MR 16, 1023].

If $r(z)$ is given by (iii), the differential equation (ii) may be given the form (iv): $d^2w/dz^2=[uz^{-1}+\gamma z^{-2}+h(z)z^{-1}]w$, in which $h(z)$ is analytic. The major portion of the paper is devoted to the derivation of asymptotic solutions of (iv) and the proof of the fact that segments of these séries are asymptotic representations of actual solutions. The domain of $z$ considered need not be bounded. As may be expected, in the case of (iv) the function $\chi(z\psi(u))$ takes the form $z^{\frac{1}{2}}C_\mu(\sqrt{4uz})$, where $C_\mu$ is a cylinder function of order $\mu=(1+4\gamma)^{\frac{1}{2}}$. The asymptotic solutions of a class of equations containing (iv) as a special case have been previously determined by A. D. Ziebur [Thesis, Univ. of Wisconsin, 1950] for $z$ confined to a bounded neighborhood of $z=0$.

In an appendix it is remarked that the hypotheses placed upon the behavior of $r(z)$ at $\infty$ in the author's paper cited above are unnecessarily severe, and they are relaxed. The possibility of using cylinder functions whose argument is an asymptotic series is also touched upon.

*N. D. Kazarinoff* (Ann Arbor, Mich.).

**Bogdanov, Yu. S. Remarks on § 81 of I. G. Malkin's monograph "Theory of stability of motion".** Prikl. Mat. Meh. 20 (1956), 448. (Russian)

Take a system in an $n$-vector $x$

$$(1)\qquad \dot{x}=P(t)x,$$

where $P(t)$ is continuous and bounded for $t\geq 0$. Let $X(t, \tau)$ be a matrix of solutions such that $X(\tau, \tau)=1$. Let $\mu_i$ denote the Lyapunov number of $x_{ii}(t, 0), \cdots$. Suppose that to every positive $\gamma$ there correspond a positive $C_\gamma$ such that whatever $t, \tau$:

$$(2)\qquad |x_{ij}(t,\tau)|<\begin{cases} C_\gamma \exp[(\gamma-\mu_i)(t-\tau)], & \text{for } 0\leq\tau\leq t,\\ C \exp[(\gamma+\mu_i)(\tau-t)], & \text{for } 0\leq t\leq\tau.\end{cases}$$

According to Malkin if (1) is regular and (2) holds then the Lyapunov characteristic numbers are stable. The author shows that conditions (2) implies regularity.

*S. Lefschetz* (Princeton, N.J.).

**Langer, Rudolph E. On the construction of related differential equations.** Trans. Amer. Math. Soc. 81 (1956), 394–410.

The most powerful known method for the asymptotic study of analytic differential equations of the form

$$(1)\quad \mathcal{L}(u)\equiv\frac{d^nu}{dz^n}+\lambda p_1(z,\lambda)\frac{d^{n-1}u}{dz^{n-1}}+\cdots+\lambda^n p_n(z,\lambda)u=0$$

is based on comparison with a "related equation", i.e., a differential equation $\mathcal{L}^*(u)=0$ of the same type as (1) satisfying the following conditions: 1) The coefficients of $\mathcal{L}^*(u)$ differ from those of $\mathcal{L}(u)$ by terms of order $O(\lambda^{-r-1})$, $r\geq 0$; 2) The asymptotic form of the solutions of $\mathcal{L}^*(u)=0$ is known in the region under consideration. The asymptotic theory of (1) is particularly important — and particularly difficult — in the neighborhood of a "turning point", i.e., a point where the multiplicities of the roots $\chi_j=\chi_j(z)$ of the auxiliary equation

$$\chi^n+p_1(z,\infty)\chi^{n-1}+\cdots+p_n(z,\infty)=0$$

change. There exists no fully general method for the construction of a related equation in the neighborhood of a turning point. The author describes a procedure for finding a related equation under the following assumption: Let $\chi_1, \cdots, \chi_{n-m}$ be the roots of the auxiliary equation that are simple at the turning point. Then there exists a differential equation $L^*(u)=0$ of order $m$ whose auxiliary equation has $\chi_{n-m+1}, \cdots, \chi_n$ as its roots and whose asymptotic properties are known.

Only a very sketchy description of the author's procedure can be given. He begins by factoring $\mathcal{L}(u)$ approximately, i.e., to within terms of order $O(\lambda^{-r-1})$, $r$ arbitrary — into a differential expression $L(U)$ of order $m$ having $\chi_{n-m+1}, \cdots, \chi_n$ as roots of its auxiliary equation and into an expression of the form $\mathcal{D}_{n-m}(u)$, where

$$\mathcal{D}_k=\prod_{j=1}^{k}\left(1-\lambda\sigma_j(z,\lambda)\frac{d}{dz}\right).$$

He then constructs the differential equation

$$(2)\qquad \begin{vmatrix} L^*(u_1) & \cdots & L^*(u_{n-m}) & L^*(u)\\ \mathcal{D}_1L^*(u_1) & \cdots & \mathcal{D}_1L^*(u_{n-m}) & \mathcal{D}_1L^*(u)\\ \cdot & \cdot & \cdot & \cdot\\ \mathcal{D}_{n-m}L^*(u_1) & \cdots & \mathcal{D}_{n-m}L^*(u_{n-m}) & \mathcal{D}_{n-m}L^*(u)\end{vmatrix}=0$$

in which $u_1, \cdots, u_{n-m}$ are certain suitably defined functions of the form

$$u_i(z,\lambda)=\exp\left(\lambda\int\sigma_j(z,\lambda)dz\right)\sum_{j=0}^{\infty}\frac{\alpha_{i,j}(z)}{\lambda_j}\quad (i=1,\cdots,n-m).$$

These functions $u_i$ together with a fundamental system $u_{n-m+1}, \cdots, u_n$ of $L^*(u)=0$ constitute a fundamental system of solutions for (2). The related equation $\mathcal{L}^*(u)=0$ is obtained by normalizing (2) so as to make its leading coefficient equal to unity.

*W. Wasow.*

**Cooke, K. L. A non-local existence theorem for systems of ordinary differential equations.** Rend. Circ. Mat. Palermo (2) 4 (1955), 301–308 (1956).

The author proves two theorems which extend Wintner's results [Amer. J. Math. 68 (1946), 173–178, 451–454; MR 7, 297; 8, 71). The first is concerned with the nonlocal existence of solutions of the system $dx/dt=f(t, x)$ and the second with the "asymptotic equilibria". The method of proofs is that of Wintner.

*M. Zlámal* (Brno).

**Wasow, Wolfgang. Singular perturbations of boundary value problems for nonlinear differential equations of the second order.** Comm. Pure Appl. Math. 9 (1956), 93–113.

A two-point boundary value problem for the real differential equation $\varepsilon y''=F(x, y, y', \varepsilon)$ is considered for

small $\varepsilon$ where $F$ is linear in $y'$. The author shows that this equation can be transformed to the form

$$\varepsilon y''+p(x,\varepsilon)y'+q(x,\varepsilon)y=\varepsilon a(x,\varepsilon)+f(x,y,\varepsilon)yy'+g(x,y,\varepsilon)y^2$$

where $p(x,0)=1$ and the boundary problem consists of finding a solution satisfying $y(0,\varepsilon)=y^0$, $y(1,\varepsilon)=0$, where $y^0$ is a constant. The solution should tend to zero for $0<x\leqq 1$ as $\varepsilon\to 0$. This problem is solved by an explicit procedure which gives the solution in terms of convergent and asymptotic series. *N. Levinson* (Cambridge, Mass.).

**Hartman, Philip; and Wintner, Aurel. On a problem of Poincaré concerning Riccati's equation.** Amer. J. Math. **77** (1955), 791–804.

Let $x(t)$, $y(t)$ be independent real solutions of $x''+f(t)x=0$, where $f(t)$ is real and continuous; let $z=x+iy$ and $w=z'/z$. The validity of the assertion $A_\omega$: $w(t)\to\omega\geqq 0$ as $t\to\infty$ is discussed. It is shown that "$f(t)\to\omega$ as $t\to\infty$" is not sufficient, and that $B_\omega$:

$$\limsup_{t\to\infty\ s>0}\left|\int_t^{t+s}(f(u)-\omega)du\right|/(1+s)=0$$

is necessary, for the truth of $A_\omega$; also $A_0$ holds either for no $w(t)$ or for all $w(t)$, whereas $A_\omega$ $(\omega>0)$ can hold for at most one $w(t)$. Further results include: $A_0$ is equivalent to "$B_0$, and $|w(t)|\to\infty$"; $A_\omega$ $(\omega>0)$ is equivalent to "$B_\omega$, and $|w(t)|$ has a finite limit". *G. E. H. Reuter.*

**Basov, V. P. Investigation of the behavior of the solutions of systems of linear differential equations in the neighborhood of an irregular point.** Ukrain. Mat. Ž. **8** (1956), 97–109. (Russian)

The author first gives a brief review of the theory of the behavior of the solutions of a system of linear differential equations near an irregular singular point. Following the general method of Erugin he then proves the following theorem. Consider the system of matrix differential equations:

$$\frac{dX_1}{dt}=t^\beta(P+Q)X_1+t^{-\alpha}R_{12}X_2,$$

$$\frac{dX_2}{dt}=t^\beta R_{21}X_1+t^{-\alpha}R_{22}X_2.$$

Here $P$ is a $m_1\times m_1$ square constant matrix whose characteristic roots have non-positive real parts, $Q$ is a $m_1\times m_1$ matrix which is continuous for real values of $t$ and such that $Q\to 0$ as $t\to+\infty$, and $R_{s\sigma}$ are $m_s\times m_\sigma$ $(s,\sigma=1,2)$ continuous and bounded matrices for real $t\geqq t^*$.

If $\alpha>1$, $\beta>-1$ then the differential system has $m_2$-columned matrix solutions of the form

$$X_1=t^{-(\alpha+\beta)}U_1,\ X_2=I+t^{-(\alpha-1)}U_2,$$

where $U_i$ $(i=1,2)$ are $m_i$-rowed matrices which are bounded for $t\geqq t^*$. *L. Markus* (Princeton, N.J.).

**Rapoport, I. M. Remarks on V. P. Basov's paper, "Investigation of the behavior of solutions of systems of linear differential equations in the neighborhood of an irregular point".** Ukrain. Mat. Ž. **8** (1956), 110–111. (Russian)

The author compares the results of Basov reviewed above with those of earlier works. *L. Markus.*

**Horošilov, V. V. On solutions of systems of linear differential equations with an irregular singular point.** Leningrad. Gos. Univ. Uč. Zap. **137.** Ser. Mat. Nauk **19** (1950), 180–197. (Russian)

The author considers the differential system $dX/dt=XP$, where $P=P_0+P_1(1/t)$. Here $P_0$ is a complex constant matrix with characteristic roots having distinct real parts and $P_1(1/t)$ is a complex analytic matrix for the complex variable $t$ near $\infty$. Thus $t=\infty$ is an irregular singular point.

It is proved that there is a solution basis of functions each represented by a power series in $1/t$, uniformly convergent on the real line near $t=+\infty$, multiplied by an exponential and a power of $t$. Also there is a basis of solutions represented by an asymptotic series multiplied by an exponential and a power of $t$.

The author remarks that the positive $t$-axis can be replaced by certain other rays in the complex plane and also that corresponding results hold for a single $n$th order equation which yields a system of the type described above. *L. Markus* (Princeton, N.J.).

**André, Johannes; und Seibert, Peter. Über stückweise lineare Differentialgleichungen, die bei Regelungsproblemen auftreten. I.** Arch. Math. **7** (1956), 148–156.

As a step toward a general theory of on-off control systems, the authors consider the system of differential equations which can be written in matrix form as follows: $\dot z=Dz+R\ \mathrm{sgn}\ Sz$. $z$, $D$, $R$, $S$ are $n\times 1$, $n\times n$, $n\times s$, $s\times n$ matrices, respectively; the elements of $D$, $R$, $S$ are real and constant, and the symbol sgn $Sz$ is defined componentwise in a natural way. If $s_1,\cdots,s_s$ denote the row-vectors of $S$, the hyperplanes represented by the equations $s_iz=0$ $(i=1,\cdots,s)$ are called switching-spaces (Schalträume). Conditions are given under which an arc of a trajectory lies in a switching-space, and such cases are thereafter excluded from consideration. If $z_0$ is a point belonging to precisely one switching-space, the arcs of trajectories in the neighborhood of $z_0$ may have various forms. The chief result of the paper is a classification of the possible forms, and a description of them by means of a complicated set-theoretic notation.

*L. A. MacColl* (New York, N.Y.).

**André, Johannes; und Seibert, Peter. Über stückweise lineare Differentialgleichungen, die bei Regelungsproblemen auftreten. II.** Arch. Math. **7** (1956), 157–164.

This is a continuation of the work begun in the paper reviewed above. The authors consider cases in which there is switching-delay, which may be of either one of two types, called, respectively, time lag and threshold. Under these conditions there may occur certain oscillations which are absent in the ideal case considered in the first paper. In the present paper these oscillations are discussed, too briefly for complete clarity, from a very general point of view. *L. A. MacColl.*

**Moser, Jürgen. Stabilitätsverhalten kanonischer Differentialgleichungssysteme.** Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. IIa. **1955**, 87–120.

Detailed results are given for stability of periodic solutions of canonical systems of one degree of freedom. The Hamiltonian $H(x,y,t)$ is assumed analytic in $x$, $y$ near $(0,0)$ and periodic in $t$ of period $2\pi$, its series expansion beginning with $-\frac{1}{2}\omega(x^2+y^2)$, where $\omega$ is a constant. A certain number $\alpha_4$ is defined, invariant under analytic canonical transformations. It is shown that, if $\alpha_4\neq 0$ and neither $3\omega$ nor $4\omega$ is an integer, then constants $c$, $N$ can be found such that for every solution $x(t)$, $y(t)$ of the differential equations, the quantity $p=(x^2+y^2)^{-\frac{1}{2}}$ satisfies the inequality $p(t)\leqq 5p(0)+c^n|t|^{1/n}$, provided $c^n\leqq p(0)<\infty$ and $n>N$. This is interpreted as implying an "almost-

stability" of the solution $x=0$, $y=0$. The result is applied to the restricted problem of three bodies; in particular, a convincing explanation is given for the famous gaps in the periods of the asteroids and a suggestion is given of a similar explanation of the spacing of the rings of Saturn.

*W. Kaplan* (Ann Arbor, Mich.).

**Moiseev, N. N. On approximate integration of linear differential equations of 2nd order.** Rostov. Gos. Univ. Uč. Zap. Fiz.-Mat. Fak. 18 (1953), no. 3, 83–98. (Russian)

In the differential equation

$$y''+\varphi(t)y=\psi(t)$$

if $\varphi(t)$ is large and positive the solution $y$ oscillates rapidly, making the usual type of direct numerical solution difficult. The author uses methods similar to the variation of parameters to derive expressions for the solutions in terms of sines and cosines and involving functions that do not vary so rapidly. The treatment is entirely analytical; the results however involve integrals which doubtless require numerical evaluation. *W. E. Milne.*

**Westman, A. E. R.; and DeLury, D. B. The differential equations of consecutive reactions.** Canad. J. Chem. 34 (1956), 1134–1138.

Das Problem der Lösung des Systems der Differentialgleichungen von aufeinander folgenden monomolekularen Reaktionen ist, wie das auch die Verfasser hervorheben, eine schon längst gelöste Frage. [Mathematisch ähnliche, wenn auch etwas einfachere Probleme treten übrigens auch in der Theorie des radioaktiven Zerfalls auf. Vgl. z.B. F. Bitter, Nuclear physics, Addison-Wesley, Cambridge, Mass., 1950, S. 141–144.] Die Lösungen werden jedoch, besonders angefangen von der, die sich auf die dritte entstehende Substanz bezieht, recht verwickelt. Die Verfasser schreiben deshalb die Lösungen der ersten und zweiten Differentialgleichung in einer analytisch etwas anderen Form, aus der sie dann eine Art Matrizenschema herleiten, aus dem man die Form der Lösungen von allen weiteren Differentialgleichungen ohne Mühe ablesen kann. Die erhaltenen Resultate werden für den Fall verallgemeinert, dass eine fragliche Substanz nicht nur aus der unmittelbar vorangehenden, sondern aus allen vorangehenden entsteht und einige Beispiele werden besprochen. *Th. Neugebauer* (Budapest).

**Moser, Jürgen. Nonexistence of integrals for canonical systems of differential equations.** Comm. Pure Appl. Math. 8 (1955), 409–436.

The author considers a canonical system $\dot{x}=H_y$, $\dot{y}=-H_x$, where $H(x, y, t)=\beta(x^2+y^2)+\sum H_{kl}(t)x^ky^l$ for $|x|+|y|<A$, the summation beginning with terms of degree three. It is assumed that $H(x, y, t)$ has period $2\pi$ in $t$. It is shown that $H$ can be arbitrarily closely approximated by a function $H^*$ such that the corresponding canonical system has isolated periodic solutions in every neighborhood of the equilibrium point $x=0$, $y=0$, and hence can have no differentiable integral of form $x^2+y^2+F(x, y, t)$, where $xF_x+yF_y=o(x^2+y^2)$ in a neighborhood of (0, 0). In the course of the proof an analogous theorem is proved for area-preserving mappings. The results supplement the classical theorems of Bruns and Poincaré for the three-body problem and a more recent theorem of Siegel [Ann. of Math. (2) 42 (1941), 806–822; MR 3, 214]. *W. Kaplan* (Ann Arbor, Mich.).

**Urabe, Minoru; and Sibuya, Yasutaka. On center of higher dimensions.** J. Sci. Hiroshima Univ. Ser. A. 19 (1955), 87–100.

Let $\dot{x}=\xi(x)$ be an analytic system, $x$, $\xi$ being $n$-vectors, $\xi(0)=0$, and assume that any solution in the neighborhood of the origin is periodic, the period being bounded. The characteristic roots of the Jacobian matrix of $\xi$ at the origin are then purely imaginary or zero and relatively commensurable, the elementary divisors being linear. If this Jacobian is not the zero-matrix, a period $\omega(x)$ of the solution passing through the point $x$ can be found such that $\omega$ is analytic. In any case where there is such an analytic period, there is an analytic change of variables $y=F(x)$ such that the new system becomes $\dot{y}_k=-[2\Theta_k\pi/\bar{\omega}(y)]y_{-k}$, $\dot{y}_{-k}=[2\Theta_k\pi/\bar{\omega}(y)]y_k$, $\dot{y}_p=0$, where $y_{\pm k}$, $y_p$ are the components of $y$, $\Theta_k$ integers, $\bar{\omega}$ analytic.

*J. L. Massera* (Montevideo).

**Ura, Taro. Sur les périodes fondamentales de solutions périodiques.** Comment. Math. Univ. St. Paul. 4 (1955), 113–130.

Let $\dot{x}=X(x)$ be a system defined in a region $A$, $x$, $X$ $n$-vectors, $X$ continuous; assume that through each point $x$ there is a unique solution $x(t)$ such that $x(0)=x$. Let $M$ be the set of points $x$ such that $x(t)$ is periodic and denote by $\omega(x)$ the corresponding fundamental period. Among other results, the following theorems are proved: 1. if $x_n\to x$, $x_n\in M$, $x$ non-singular, $\liminf \omega(x_n)=\omega_0<\infty$, then $\omega_0>0$ and, if $x(\omega_0)$ is defined, $x\in M$ and $\omega_0/\omega(x)$ is an integer; consequently $\omega$ is lower semicontinuous in $M$, the set of points where $\omega$ is continuous is open in $M$. 2. If $\omega$ is locally bounded in $M$, the set of points where $\omega$ is discontinuous is non-dense (the assumption of boundedness is essential). 3. If $x_n\to x$, $x_n\in M$, $\omega(x_n)$ bounded, then either $x(t)$ approaches the boundary of $A$ for a certain positive and a certain negative value of $t$, or the trajectory $x(t)$ is a bounded point set to which the sequence $x_n(t)$ converges uniformly; the last alternative takes place, in particular, if $x$ is a singular point. *J. L. Massera.*

**Lewis, D. C. On the role of first integrals in the perturbation of periodic solutions.** Ann. of Math. (2) 63 (1956), 535–548.

After a number of Lemmas which are interesting in themselves, the following main result is proved. Let (3.1) $\dot{x}=A(t)x+f(x, t, \mu)$, where $x$, $f$ are $n$-vectors, $A$ continuous matrix, $f$ of class $C'$ for $\|x\|\leqq r$, $-\infty<t<+\infty$, $|\mu|<\lambda$, $f(0, t, 0)=f_x(0, t, 0)=0$, $A$, $f$ periodic in $t$ of period $T$. Assume that $\dot{x}=Ax$ has $k$ linearly independent periodic solutions and that (3.1) has $k$ independent (Jacobian matrix of rank $k$) first integrals of class $C''$. Then if $\|c\|$, $|\mu|$ are sufficiently small there exists a unique function $x(t, c, \mu)$ of class $C'$ which is a periodic solution of (3.1), $c$ being a $k$-vector parameter. In the analytic case numbers $h$, $L$ can be explicitly constructed so that the above mentioned result holds when $\|c\|<h$, $|\mu|<L$.

*J. L. Massera* (Montevideo).

**Volosov, V. M. Differential equations of motion, containing a slowness parameter.** Dokl. Akad. Nauk SSSR (N.S.) 106 (1956), 7–10. (Russian)

Consider

$$(1)\qquad d[m(\mu t)\dot{x}]/dt+\mu f(\mu t, x, \dot{x})+Q(\mu t, x)=0,$$

$\mu$ small, $Q$ is twice continuously differentiable in $0\leqq\mu t\leqq a$, $|x|\leqq\sigma$, $\operatorname{sgn} Q=\operatorname{sgn} x$, $\partial Q/\partial x|_{x=0}\neq0$; $f$ has continuous first partials for $0\leqq\mu t\leqq a$, $-\infty<x$, $\dot{x}<+\infty$; $m$ is twice

continuously differentiable in $[0, a]$, $m>0$. Then: 1. If $x_0=\dot{x}_0=0$, the solution exists for $t \in [0, a/\mu]$ if $\mu$ is sufficiently small, $x$ and $\dot{x}$ being $O(\mu)$; 2. If $\dot{x}_0^2+x_0^2\neq 0$ and $(x_0, \dot{x}_0)$ belongs to a certain region, the solution exists for $t \in [0, \tau_0/\mu]$ if $\mu$ is sufficiently small, $\tau_0$ depending on $(x_0, \dot{x}_0)$, and oscillates in such a way that the maxima and minima lie in neighborhoods of width $k\mu$ of certain curves $x=F_1(\mu t)$, $x=F_2(\mu t)$, $F_2<0<F_1$; the "period" $T$ (time interval between two consecutive maxima or minima in the neighborhood of $t$) depends on $t$ and is of order $\mu^{\frac{1}{2}}$. The functions $F_1$ and $F_2$ are solutions of an explicit associated differential equation satisfying certain subsidiary conditions (which depend on $x_0$, $\dot{x}_0$). Formulas for the calculation of $T$ up to terms in $\mu^2$ as well as more precise estimations of the loci of the extrema are also given.

*J. L. Massera* (Montevideo).

★ **Stoker, J. J. Non-linear vibrations of systems with several degrees of freedom.** Proceedings of the Second U. S. National Congress of Applied Mechanics, Ann Arbor, 1954, pp. 33–43. American Society of Mechanical Engineers, New York, 1955. $9.00.

Review article, dealing with perturbation methods for finding periodic solution. It includes a brief survey of recent work on partial differential equations with small non-linearities, e.g. $u_{xx}-u_{tt}=\varepsilon F(u, u_t, x, t, \varepsilon)$.

*G. E. H. Reuter* (Manchester).

**Manfredi, Bianca. Sulla stabilità del moto di sistemi a più gradi di libertà in condizioni non lineari.** Boll. Un. Mat. Ital. (3) **11** (1956), 64–71.

Let (1′) $\ddot{X}+\Omega(X, \dot{X})\dot{X}+g(X)=F(t)$, where $X$ is an $n$-vector, $\Omega$ a diagonal matrix with non-negative elements, $g=\operatorname{grad} G$ with $G>0$ when $\|X\|\neq 0$, $G\to+\infty$ when $\|X\|\to\infty$ $\|F\|$ integrable in $(0, +\infty)$; then any solution is bounded in $(0, +\infty)$. Let (1″) $\ddot{X}+R(\dot{X})+\omega^2X=F(t)$, where $R\cdot\dot{X}\geqq 0$; the same conclusion holds. Let (1‴) $\ddot{X}+2\varepsilon\dot{X}+R(\dot{X})+\omega^2X=F(t)$, where $\varepsilon>0$, $R$ satisfies $\|R(X'')-R(X')\|\leqq K\|X''-X'\|^\alpha$, $\alpha<1$; then the difference between any two solutions is bounded (whence if any solution is bounded so are all of them).

*J. L. Massera* (Montevideo).

**Antosiewicz, H. A. Stable systems of differential equations with integrable perturbation term.** J. London Math. Soc. **31** (1956), 208–212.

Let $x(t)\equiv 0$ and $y(t)\equiv 0$ be the solutions of a system of equations of the form (1) $dx/dt=p(x,t)+q(x, t)$ and (2) $dy/dt=p(y, t)$ respectively. Let exist a quadratic form in $x$, $V(x, t)$, with coefficients that are continuously differentiable functions of $t$ for $t\geqq 0$ which satisfies for all $x$ and $t\geqq 0$

$$A\|x\|^2\leqq V(x, t)\leqq B\|x\|^2,$$

$$\frac{\partial V(x, t)}{\partial t}+\operatorname{grad} V(x, t)\cdot p(x, t)\leqq -C\|x\|^2,$$

where $A$, $B$, $C$ are positive constants [this condition implies the uniform asymptotic stability of $y(t)\equiv 0$ of (2); see Malkin, Prikl. Mat. Meh. **18** (1954), 129–138; MR **15**, 873]. It is proved that under the condition $\|q(x, t)\|\leqq h(t)\|x\|$ for $\|x\|\leqq\xi$, $t\geqq 0$, where $h(t)\geqq 0$ and $\int_0^\infty h(t)dt<\infty$, the solution $x(t)\equiv 0$ of (1) is also uniformly asymptotically stable. Further a kind of boundedness result is established for the system $dx/st=p(x, t)+q(x, t, z)$ where $z$ denotes a vector parameter.

*M. Zlámal* (Brno).

**Massera, José L. Contributions to stability theory.** Ann. of Math. (2) **64** (1956), 182–206.

This is a thorough study of necessary and sufficient conditions, of the type considered in Lyapunov's second second method, for the stability (simple, asymptotic, asymptotic in the large, total, in the first approximation) and the instability of the solution $x=0$ of equations $\dot{x}=f(t, x)$ with $f$ continuous on some semi-cylinder $L\times S$, $L=[0, \infty)$, $S=S(0, p)$, and $f(t, 0)\equiv 0$ on $L$. The results, most of which hold in general Banach spaces, represent important advances in stability theory.

Generalizing his earlier work [Ann. of Math. (2) **50** (1949), 705–721; MR **11**, 721], on which Malkin [Prikl. Mat. Meh. **18** (1954), 129–138; MR **15**, 873] and Barbašin and Krasovskiĭ [ibid. **18** (1954), 345–350; MR **15**, 957] based their criteria for the uniform-asymptotic stability (u.a.s) and the u.a.s. in the large of $x=0$, respectively, the author proves the following much stronger theorems: If $x \in E^n$ and $f$ is locally Lipschitzian on $L\times S$, the u.a.s. (the u.a.s. in the large) of $x=0$ implies the existence of a positive definite Lyapunov function $V(t, x)$ with $V(t,x)\to 0$ as $x\to 0$ uniformly in $t$, (and $V(t, x)\to\infty$ as $\|x\|\to\infty$ uniformly in $t$), having continuous Fréchet differentials of all orders, such that $\delta[V(t, x); (1, f(t, x)]$ is negative definite. Among the other results, too numerous to be stated in detail here, are generalizations of a theorem of Lyapunov [cf., e.g., Problème général de la stabilité du mouvement, Princeton, 1947, pp. 276–278; MR **9**, 34] on linear equations with constant coefficients, of a theorem of Četaev [Dokl. Akad. Nauk SSSR (N.S.), **1** (1934), 529–531; Uč. Zap. Kazan. Gos. Univ. **98** (1938), no. 9, 43–58] and Persidskiĭ [cf., e.g., Uspehi Mat. Nauk (N.S.) **1** (1946), no. 5-6(15–16), 250–255; MR **10**, 456] on instability, of a theorem of Malkin [Dokl. Akad. Nauk SSSR (N.S.) **76** (1951), 783–784; MR **12**, 827] on stability in the first approximation, and of theorems of Malkin [Prikl. Mat. Meh. **8** (1944), 241–245; MR **7**, 298; **11**, 439] and Goršin [Izv. Akad. Nauk Kazah. SSR. **56**, Ser. Mat. Meh. **2** (1948), 46–73; MR **14**, 48] on total stability.

*H. A. Antosiewicz* (Washington, D.C.).

**Pignani, T. J.; and Whyburn, W. M. Differential systems with interface and general boundary conditions.** J. Elisha Mitchell Sci. Soc. **72** (1956), 1–14.

In the first part the authors study the system

$$Y'+P(x)Y=Q(x), \tag{1}$$

$$A_rY(t_r^+)+B_rY(t_r^-)=C_r\ (r=1, 2, \cdots, m-1), \tag{2}$$

$$AY(a)+BY(b)+\int_a^b F(s)Y(s)ds=D. \tag{3}$$

Capital letters in (1), (2), (3) denote $(n\times n)$ matrices; $A_r$, $B_r$, $C_r$, $A$, $B$ and $D$ are matrices of constants, while $P(x)$, $Q(x)$ and $F(x)$ are Lebesgue summable matrix functions of the real variable $x$ on the interval $[a, b]$. (2) are interface conditions. The interface points $t_1$, $t_2$, $\cdots$, $t_{m-1}$ are such that $a=t_0<t_1<t_2<\cdots<t_{m-1}<t_m=b$ and $Y(t_r^-)$, $Y(t_r^+)$ designate the limits of $Y(t)$ as $t$ approaches the point $t_r$ from the left and right, respectively. A $d$-solution of (1) and (2) on $[a, b]$ is a matrix, $Y(x)$, which is absolutely continuous on $(t_{i-1}, t_i)$ $(i=1, 2, \cdots, m)$, satisfies (1) almost everywhere on $[a, b]$ and satisfies (2). (3) are general boundary conditions. The results of the authors are extensions of those of Stallard [Oak Ridge Nat. Lab. Rep. ORNL **1876** (1955)]. The second part is concerned with the same system with the exception that the number of interface points may be countably infinite. *M. Zlámal.*

See also: Turán, p. 4; Koutský, p. 30; Doetsch, p. 35; Levin, p. 35; Wiebelitz, p. 49; Bompiani, p. 65; Fettis, p. 73; Mikeladze, p. 73; Giger, p. 73; Punnis, p. 88.

## *Partial Differential Equations*

Have

★ **Friedman, Bernard. Principles and techniques of applied mathematics.** John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London, 1956. ix+315 pp. $8.00.

Cet ouvrage est principalement consacré aux fonctions de Green des opérateurs différentiels et aux dérivées partielles, pour différents problèmes aux limites. L'auteur étudie leurs propriétés, comment les obtenir, la manière d'exprimer la solution des problèmes aux limites au moyen de ces fonctions et les relations de ces dernières avec la représentation spectrale de la solution des problèmes considérées. Il se borne d'ailleurs à traiter en détails les problèmes aux limites relatifs aux opérateurs différentiels

$$a(x)(d^2/dx^2)+b(x)(d/dx)+c(x),\ a,\ b,\ c\in C_0([0,1]),$$

avec des conditions aux limites dans [0, 1] du type

$$\alpha_{10}u(0)+\alpha_{11}u'(0)+\beta_{10}u(1)+\beta_{11}u'(1),$$
$$\alpha_{20}u(0)+\alpha_{21}u'(0)+\beta_{20}u(1)+\beta_{22}u'(1),$$

et, plus sommairement, certains problèmes aux limites pour les opérateurs aux dérivées partielles

$$\Delta,\ \Delta-\frac{\partial}{\partial t},\quad \Delta-\frac{\partial^2}{\partial t^2},\quad \Delta=\sum_1^n\frac{\partial^2}{\partial x_i^2}.$$

Un chapitre introductif concerne l'étude des espaces linéaires.

{Il s'agit d'un ouvrage destiné à montrer l'intérêt de la théorie abstraite des espaces linéaires dans l'unification et la systématisation des techniques de la mathématique appliquée. Signalons que le livre, en dépit de son titre, ne couvre qu'une partie assez limitée des mathématiques appliquées. D'autre part, l'exposé est souvent heuristique. Certaines considérations ne sont fondées que sur des exemples et sur l'analogie avec ce qui se passe pour les systèmes algébriques linéaires. La fonction de Dirac est utilisée systématiquement (l'auteur la définit cependant en faisant appel à la théorie des distributions mais ne semble pas justifier complètement les emplois ultérieurs qu'il en fait). De ce fait, la lecture de cet ouvrage laisse un arrière-goût d'imprécision assez gênant, particulièrement marqué dans les derniers chapitres.

Ce livre peut néanmoins ouvrir des horizons en présentant sous un jour nouveau certaines questions classiques. Cependant, le reviewer se demande si c'est vraiment rendre service à ceux qui appliquent les mathématiques aux problèmes concrets que de les habituer à des raisonnements incomplets ou heuristiques ne pouvant entraîner qu'une connaissance formelle, superficielle et partant dangereuse, des questions traitées. Ne vaudrait-il pas mieux présenter les théories de l'analyse abstraite d'une manière simple, complété et rigoureuse, dans le cadre des problèmes particuliers étudiés?}

Table des matières: 1) Linear spaces (principalement espaces de Hilbert); 2) Spectral theory of operators (réduite à une belle étude de la réduction des matrices à la forme canonique de C. Jordan plus quelques indications sur les opérateurs définis dans les espaces linéaires); 3) Green's functions (des opérateurs différentiels); 4) Eigenvalue problems of ordinary différential operators; 5) Partial differential equations.

La présentation et la typographie sont excellentes.

*H. G. Garnir* (Liège).

**Keller, J. B.; Lewis, R. M.; and Seckler, B. D. Asymptotic solution of some diffraction problems.** Comm. Pure Appl. Math. 9 (1956), 207–265.

This is identical with Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **EM-81** (1955) [MR 17, 41].

*E. T. Copson* (St. Andrews).

**Filin, A. P. On a direct method of solution of a boundary problem.** Inžen. Sb. 22 (1955), 53–64. (Russian)

The solution of various boundary value problems for partial differential equations, or for systems of partial differential equations, is approached as a problem in determining the coefficients in an orthogonal expansion of the solution with respect to some complete set of functions. For convenience, two or more orthogonal expansions are combined, so that (for example), the solution $W$ is given by

$$W=\sum_{i=1}^{\infty}a_iW_i+\sum_{m,n=1}^{\infty}b_{mn}W_{mn}^0,$$

where the functions $W_i$ satisfy the given homogeneous boundary conditions, and the coefficients $a_i$ are determined from the given inhomogeneous boundary conditions. The functions $W_{mn}^0$, on the other hand, satisfy homogeneous boundary conditions in all cases, and the coefficients $b_{mn}$ are determined from the partial differential equation itself. The method is applied to a variety of examples, including a hinged and loaded rectangular plate, and a three dimensional problem in elasticity. The determination of the coefficients involves, in general, the solution of an infinite system of simultaneous linear algebraic equations by means of determinants of infinite order.

*R. B. Davis* (Syracuse, N.Y.).

**Rachajsky, B. Intégrales de S. Lie et les transformations de contact.** Bull. Soc. Math. Phys. Serbie 6 (1954), 162–171. (Serbo-Croatian. French summary)

Verfasser geht von den Gleichungen einer allgemeinen Berührungstransformation $q$ter Klasse (und ihren inversen) aus und löst die Aufgabe, die partielle Differentialgleichung erster Ordnung

$$\Phi(x_1,\cdots,x_n,z,p_1,\cdots,p_n)=0$$

zu ermitteln, welche durch die Berührungstransformation in die Funktionalgleichung

$$F(x_1',\cdots,x_n',z')=0$$

verwandelt wird. Auch die umgekehrte Aufgabe, die Bestimmung der entsprechenden Berührungstransformation der Klasse $q$ erweist sich angreifbar und wird von Verfasser auf die Behandlung eines verallgemeinerten Charpit'schen Systems für die charakteristischen Funktionen der Berührungstransformation zurückgeführt. Methodisch stützt sich dabei Verfasser auf seine vorhergehenden Arbeiten [dasselbe Bull. 5 (1953), no. 3–4, 79–90; Acad. Roy. Belg. Bull. Cl. Sci. (5) 40 (1954), 896–909; MR 16, 252, 479] denen grundlegende Arbeiten von S. Lie [Math. Ann. 9 (1876), 245–296] und N. Saltykow [J. Math. Pures Appl. (5) 3 (1897), 423–428; C. R. Acad. Sci. Paris 137 (1903), 309–312; Atti IV Congresso Internaz. Mat., Roma, 1908, v. 2, Accad. Lincei, Roma, 1909, pp. 77–86; Acad. Roy. Serbe. Bull. Acad. Sci. Math. Nat. A. 5 (1939), 121–

137; Srpska Akad. Nauka. Posebna Izdanja **139**, Prirod. Mat. Spisi **38** (1947), § 43, 204; MR **1**, 313; **10**, 253] vorausgehen. *M. Pinl* (Köln).

**Aržanyh, I. S. Universal significance of contact transformations.** Uspehi Mat. Nauk (N.S.) **11** (1956), no. 1(67), 167–172. (Russian)

Verfasser schreibt vektoriell $p, q, \xi, \eta, f, g, \cdots$ an Stelle von je $n$ Komponenten $p_\gamma, q_\gamma, \xi_\gamma, \eta_\gamma, f_\gamma, g_\gamma, \cdots$ und benutzt ferner Abkürzungen wie

$$\frac{\partial W}{\partial q}, \frac{\partial W}{\partial \xi}, \frac{\partial W_\kappa}{\partial q}, \frac{\partial W_\kappa}{\partial \xi}, \frac{\partial H_\sigma}{\partial p}, \frac{\partial H_{\iota\sigma}}{\partial q}, \cdots, \frac{\partial H_\sigma}{\partial q}, \frac{H_{\iota\sigma}}{\partial p}, \cdots$$

an Stelle von

$$\frac{\partial W}{\partial q_\gamma}, \frac{\partial W}{\partial \xi_\gamma}, \frac{\partial W_\kappa}{\partial q_\gamma}, \frac{\partial W_\kappa}{\partial \xi_\gamma}, \frac{\partial H_\sigma}{\partial p_\gamma}, \frac{\partial H_{\iota\sigma}}{\partial q_\gamma}, \frac{\partial H_\sigma}{\partial q_\gamma}, \frac{\partial H_{\iota\sigma}}{\partial p_\gamma}, \cdots \qquad (\gamma=1, 2, \cdots, n).$$

Dann wird gezeigt: das Pfaffsche System

$$(*)\qquad \begin{aligned} dq&=\Big(\frac{\partial H_\sigma}{\partial p}+F_{\iota\sigma}\frac{\partial H_{\iota\sigma}}{\partial p}\Big)dt_\sigma,\\ -dp&=\Big(\frac{\partial H_\sigma}{\partial q}+F_{\iota\sigma}\frac{\partial H_{\iota\sigma}}{\partial q}\Big)dt_\sigma, \end{aligned} \qquad (\sigma=1, 2, \cdots, s).$$

verwandelt sich durch die Berührungstransformation

$$p=\frac{\partial W}{\partial q}+\Lambda_\kappa\frac{\partial W_\kappa}{\partial q}, -\eta=\frac{\partial W}{\partial \xi}+\Lambda_\kappa\frac{\partial W_\kappa}{\partial \xi}$$

in das Pfaffsche System

$$(**)\qquad \begin{aligned} d\xi&=\Big(\frac{\partial K_\sigma}{\partial \eta}+F_{\iota\sigma}\frac{\partial K_{\iota\sigma}}{\partial \eta}\Big)dt_\sigma,\\ -d\eta&=\Big(\frac{\partial K_\sigma}{\partial \xi}+F_{\iota\sigma}\frac{\partial K_{\iota\sigma}}{\partial \xi}\Big)dt_\sigma, \end{aligned}$$

mit

$$K_\sigma=H_\sigma+\frac{\partial W}{dt_\sigma}+\frac{\partial W_\kappa}{\partial t_\sigma}\Lambda_\kappa.$$

Dabei sind $W$ und $W_\kappa$ als Funktionen von $t_1, t_2, \cdots, t_s; q; \xi$ gegeben; die Faktoren $\Lambda_\kappa$ bestimmen sich aus den Gleichungen

$$W_\kappa(t_1, t_2, \cdots, t_s; q; \xi)=0 \quad (\kappa=1, 2, \cdots, k; \sigma=1, 2, \cdots, s).$$

Das System (*) verhält sich invariant gegenüber Berührungstransformationen der Gestalt:

$$q=u-\frac{\partial \Phi}{\partial v}-M_\lambda\frac{\partial \Phi_\lambda}{\partial v}, \quad \xi=u+\frac{\partial \Phi}{\partial v}+M_\lambda\frac{\partial \Phi_\lambda}{\partial v},$$

$$p=v+\frac{\partial \Phi}{\partial u}+M_\lambda\frac{\partial \Phi_\lambda}{\partial u}, \quad \eta=v-\frac{\partial \Phi}{\partial n}-M_\lambda\frac{\partial \Phi_\lambda}{\partial u}.$$

Dabei sind $\Phi$ und $\Phi_\kappa$ als Funktionen von $t_1, \cdots, t_s; u, v$ gegeben; die Faktoren $M_\lambda$ bestimmen sich aus den Gleichungen

$$\Phi_\lambda(t_1, \cdots, t_s; u, v)=0 \quad (\lambda=1, 2, \cdots, l).$$

Für das so transformierte System (**) gilt

$$K_\sigma=H_\sigma+2\Big(\frac{\partial \Phi}{\partial t_\sigma}+M_\lambda\frac{\partial \Phi_\lambda}{\partial t_\sigma}\Big), \quad K_{\iota\sigma}=H_{\iota\sigma}.$$

Zum Schluß betrachtet Verfasser ein vollständig integrables System der Gestalt:

$$(***)\qquad \begin{aligned} \Omega(d)&\equiv A dq+(B-\chi I)dp-I(A d\xi-(B+\chi I)d\eta)=0,\\ \Theta(d)&=(B+\chi I)dq-C dp-I((B-\chi I)d\xi+C d\eta)=0. \end{aligned}$$

Die Matrizen $A$, $B$, $C$ sollen symmetrisch von den Vektoren $p$, $q$, $\xi$, $\eta$ abhängen und das Quadrat der Matrix $I$ vermehrt um die Einheitsmatrix I soll verschwinden. Dann lassen die durch (***) bestimmten Berührungstransformationen die Struktur des Systems (*) ungeändert. Da jedes partielle Differentialsystem (abgesehen von algebraischen Zusatzbedingungen) stets durch ein Pfaffsches System dargestellt werden kann, überträgt sich die Bedeutung der Berührungstransformationen für die Integration Pfaffscher Systeme auf die Integration der äquivalenten partiellen Systeme, sofern deren Kompatibilität gesichert ist. Damit ist die Bedeutung der Berührungstransformationen für die allgemeine Integrationstheorie beträchtlich erweitert. Insbesondere gilt dies auch für Systeme, welche mit wirbelartigen "Impulstermen" behaftet sind. Derartige Terme $\partial p_\nu/pq_\mu-\partial p_\mu/\partial q_\nu$ werden wieder in wirbelartige Terme $\partial\eta_\nu/\partial\xi_\mu-\partial\eta_\nu/\partial\xi_\mu$ verwandelt, so daß die Invarianz dieses Prozesses prinzipielle Bedeutung für die theoretische Physik gewinnt.
*M. Pinl* (Köln).

**Reid, Walter P. A method for solving certain boundary value problems.** J. Soc. Indust. Appl. Math. **3** (1955), 259–261.

Die Methode wird zunächst am folgenden Beispiel erläutert.

$$u_{xx}+u_{yy}=0 \quad (u \text{ beschränkt für } x>0, y>0),$$
$$a_0u+a_1u_x+\cdots+a_nu_{x^n}=f(y) \quad \text{für } x=0,$$
$$b_0u+b_1u_y+\cdots+b_ku_{y^k}=g(x) \quad \text{für } y=0,$$

wo $f(y)$ und $g(x)$ für alle $y$ und $x$ bzw. beschränkt sind. Macht man den Ansatz

$$w=\Big(a_0+a_1\frac{\partial}{\partial x}+\cdots+a_n\frac{\partial^n}{\partial x^n}\Big)\Big(b_0+b_1\frac{\partial}{\partial y}+\cdots+b_k\frac{\partial^k}{\partial y^k}\Big)u$$

so ergibt sich für $w$ dieselbe Differentialgleichung mit Randbedingungen erster Art. Die Auffindung von $u$ erfordert die Auflösung von zwei gewöhnlichen Differentialgleichungen. Durchgeführt wird diese Methode an folgendem Beispiel $h^2u_{xx}-u_t=0$ ($u$ beschränkt für $x>0$, $t>0$), $au-u_x=0$ für $x=0$, $u=g(x)$ für $t=0$. *P. Funk.*

**Saltykow, Nicola N. Equations aux dérivées partielles du premier ordre intégrables par séparation des variables et généralisations sur l'équation de Schrödinger.** Bull. Soc. Math. Phys. Serbie **7** (1955), 137–152 (1956). (Serbo-Croatian summary)

Eine partielle Differentialgleichung erster Ordnung

$$F(x_1, x_2, \cdots, x_n, p_1, p_2, \cdots, p_n)=0 \qquad \Big(p_i=\frac{\partial z}{\partial x_i}, i=1, 2, \cdots, n\Big)$$

muß nach T. Levi Civita den Bedingungen

$$\frac{\partial(F, F_k/F_{n+k})}{\partial(x_i, p_i)}=0 \qquad (k=1, \cdots, n; i=1, 2, \cdots, k-1, k+1, \cdots, n)$$

genügen, wenn sie nach der Methode der Separation der Variablen integrierbar sein soll. Man erhält auf diese Weise $N=\frac{1}{2}n(n-1)$ unabhängige Bedingungen in der Gestalt von partiellen Differentialgleichungen zweiter Ordnung für die unbekannte Funktion $F$. Verfasser verwandelt zunächst dieses System in ein solches erster

Ordnung für die Funktion $F$ und $k$ neue Hilfsfunktionen:

$$\varphi_k=\frac{F_k}{F_{n+k}},\ F_k=\frac{\partial F}{\partial x_p},\ F_{n+k}=\frac{\partial F}{\partial p_k}\ (k=1, 2, \cdots, n).$$

Zusammen mit den die $\varphi_k$ definierenden Gleichungen ergeben sich so $\frac{1}{2}n(n+1)$ Gleichungen erster Ordnung für $n+1$ unbekannte Funktionen. Unter den Lösungen dieses (erweiterten) Systems gibt es singuläre, welche die Koeffizienten der Ableitungen $F_i$, $F_{n+i}$ zum Verschwinden bringen. Geht man von einer solchen singulären Lösung aus, setzt also die Koeffizienten gleich Null, so ergeben sich weitere Differentialgleichungen und aus diesen unmittelbar:

$$\varphi_1=f_1(x_1, p_1),\ \varphi_2=f_2(x_1, p_1; x_2, p_2),\ \cdots,$$
$$\varphi_{n-2}=f_{n-2}(x_1, p_1; x_2, p_2; \cdots; x_{n-2}, p_{n-2}),$$
$$\varphi_{n-1}=f_{n-1}(x_1, p_1; x_2, p_2; \cdots; x_{n-2}, p_{n-2}; x_{n-1}, p_{n-1})$$

mit $n-1$ willkürlichen Funktionen $f_1, f_2, \cdots, f_{n-1}$. Eine singuläre Lösung heißt von der Klasse $N-K$, wenn sie dem Verschwinden der Koeffizienten aller partiellen Ableitungen der Funktion $F$ innerhalb von $K$ beliebig aus dem (nichterweiterten) System ausgewählter Gleichungen entspricht. Ihre Anzahl ist

$$\frac{N(N-1)\cdots(N-K+1)}{K!}$$

und es gibt ebensoviele durch Separation der Variablen integrable Differentialgleichungen der Klasse $N-K$. Im Falle von $n$ unabhängigen Veränderlichen ergeben sich also

$$\Sigma_n=1+N+\frac{N(N-1)}{1\cdot 2}+\cdots$$
$$+\frac{N(N-1)\cdots(N-K+1)}{K!}+\cdots+N+1$$
$$(N=\tfrac{1}{2}n(n-1)),$$

verschiedene Typen solcher Gleichungen, z.B. für $n=2$, 3, 4, 5:

| $n$ | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| $N$ | 1 | 3 | 6 | 10 |
| $\Sigma_n$ | 3 | 8 | 64 | 1024 |

Eine ähnliche Abzählung war bereits von P. Burgatti gegeben worden [Atti Accad. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (5) **20** (1911), 108–111]. Burgatti's Zahlen stimmen mit den von Verfasser gegebenen nicht überein, was in dem weit spezielleren Charakter seiner Voraussetzungen begründet ist. Nach einer eingehenden Diskussion des Falles $n=4$ behandelt Verfasser weiterhin den Fall Hamilton-Jacobischer partieller Differentialgleichungen, für welche $F$ durch

$$H=\sum_{i,k=1}^{n} A_{ik}p_ip_k+U,\ A_{ik}=A_{ki}=A_{ik}(x_1, \cdots, x_n)$$
$$(U=U(x_1, \cdots, x_n))$$

zu ersetzen ist. Hier ergeben sich zahlreiche Beziehungen zu älteren Ergebnissen von N. Forbat und M. Stoyadinovitch.

*M. Pinl* (Köln).

**Magenes, Enrico. Sul teorema dell'alternativa nei problemi misti per le equazioni lineari ellittiche del secondo ordine.** Ann. Scuola Norm. Sup. Pisa (3) **9** (1955), 161–200 (1956).

Soit l'opérateur différentiel

$$E=\sum a_{ij}\partial^2/\partial x_i\partial x_j+\sum b_i\partial/\partial x_i+c,$$

donné dans un ouvert $\Omega$ borné de $R^n$, de frontière $F$ régulière. Les coefficients sont réels, $a_{ij}$ deux fois continûment différentiable holdériens, $b_i$ une fois continûment différentiable holdérien, $c$ holdérien; $E$ est uniformément elliptique dans $\Omega$. On donne $\mu$ de carré sommable sur une partie $F_1$ de $F$, $\delta$ de carré sommable sur $F_2=F-F_1$. Théorème: on donne $f$ holdérien de carré sommable sur $\Omega$; l'équation $Eu=f$ admet une solution unique ayant les propriétés: $u$ est deux fois continûment différentiable dans $\Omega$, $\in L^2(\Omega)$; $u=\mu$ sur $F_1$; $\partial u/\partial\nu$ (dérivée normale)$=\delta$ sur $F_2$ (ces deux égalités dans un sens généralisé). Noter que $\partial u/\partial x_i$ est localement mais non globalement $L^2$; c'est ce qui permet de prendre $\mu$ et $\delta$ dans $L^2$ (sur $F_1$ et $F_2$) ce qui n'est pas possible si l'on impose à $u$ d'être à Dirichletien fini (contre exemple de Hadamard). La démonstration repose essentiellement sur les deux points suivants: 1) Théorème d'Amerio: il existe une suite de polynomes $u_n$ tels que $Eu_n\to f$, $u_n|_{F_1}\to\mu$, $\partial u_n/\partial\nu|_{F_2}\to\delta$ dans les espaces $L^2$ convenables sous la condition

$$\int_\Omega vfdx-\int_{F_1}\mu(\partial v/\partial\nu-bv)d\sigma+\int_{F_2}\delta v d\sigma=0$$

pour tout $v$ avec $E^*v=0$ ($E^*$ étant l'opérateur adjoint de $E$), $v=0$ sur $F_1$, $\partial v/\partial\nu-bv=0$ sur $F_2$, avec $b=\sum (b_i-\sum \partial a_{ij}/\partial x_j)\cos(x_i, n)$, $n=$normale à $F$; 2) des majorations à priori (p. 173) permettent d'extraire de $u_n$ une suite convergent vers une solution.

*J. L. Lions* (Nancy).

**Beckert, Herbert. Das Dirichletsche Problem des Systems der Jacobischen Gleichungen eines zweidimensionalen Variationsproblems für $n$ gesuchte Funktionen im linearen und quasilinearen Fall.** Math. Nachr. **15** (1956), 7–29.

The author considers regular variational problems,

$$(*)\quad E(u_i, u_i)=\iint_{D+S}\sum_{i,k=1}^{n}[a_{ik}(x, y)p_ip_k+2b_{ik}(x, y)p_iq_k$$
$$+c_{ik}(x, y)q_iq_k+e_{ik}(x, y)u_iu_k+\chi_i(x, y)u_i]dxdy\to\min,$$
$$p_i=\frac{\partial u_i}{\partial x},\ q_i=\frac{\partial u_i}{\partial y},$$

for $n$ unknown functions $u_i(x, y)$ which are to vanish on the boundary $S$ of a suitable plane region $D$. The Euler-Lagrange equation corresponding to $E(u_i, u_i)$ is the Jacobi equation (equation of variation) for the solution of a variational problem depending on a parameter $\lambda$,

$$J_\lambda=\iint_{D+S}\Big\{f(x, y, p_i, q_i)+\lambda\sum_{i=1}^{n}\phi_iu_i\Big\}dxdy\to\min$$

for $n$ functions $u_i$ in $D$ assuming prescribed data on $S$.

The discussion is in the spirit of the work of Morrey and of Nirenberg, and uses also some estimates of Schauder type on the solutions of first order linear elliptic systems developed by the author. A feature is the introduction of the norm

$$N(u)=\max_{P\subset D+S}\sqrt{\Big(\iint_{D+S}\frac{u_x^2+u_y^2}{r^\alpha}dxdy\Big)},$$

where $r$ denotes distance from a point $P$ and $0<\alpha<1$. The author uses the Schauder fixed-point theorem to show that if the coefficients in (*) are bounded and measurable there exists a unique solution of (*) which is absolutely continuous in the sense of Tonelli and has finite norm. Further assumptions on the coefficients are shown to imply a corresponding improvement in the regularity of the solution.

The author includes some extensions of his results to a quasi-linear first order system which generalizes the Haar equations of (*). *R. Finn* (Pasadena, Calif.).

**Sokolov, G. T. On solutions of nonlinear equations of hyperbolic type.** Dokl. Akad. Nauk Uzbek. SSR. **1953**, no. 4, 3–7. (Russian. Uzbek summary)

The problem is

$$Lz\equiv(p(x)z_x)_x-z_{tt}=\Phi(x,t)+\mu(f(z)\ (0\leq x\leq\pi,\ 0\leq t\leq 1)$$
$$z(0,t)=0=z(\pi,t),\ z(x,0)=\varphi(x),\ z_t(x,0)=\psi(x).$$

Essential assumptions: $p(x)\geq b>0$ and $p''(x)$ is continuous and of bounded variation; $\Phi_x$ is continuous and of bounded variation in each argument, and $\Phi$ has an expansion in eigenfunctions of the problem $(pX')'+\lambda X=0$, $X(0)=0=X(\pi)$; $f(0)=0$ and $|f(z')-f(z)|<M|z'-z|$; $\varphi$ and $\psi$ are sufficiently smooth and sufficiently flat at 0 and $\pi$. Conclusion: The problem has a unique solution if $|\mu|<B$; an explicit formula is given for $B$, one factor of which is the reciprocal of sup $|f'(z)|$. Starting with the solution $z_0$ of the problem for $\mu=0$, for $k=0,1,\cdots$, $z_{k+1}$ satisfies $Lz=\Phi+\mu f(z_k)$; convergence is established from explicit formulas for the $z_k$. *F. A. Ficken.*

**Karp, V. N. On periodic solutions of a nonlinear equation of hyperbolic type.** Dokl. Akad. Nauk Uzbek. SSR. **1953**, no. 5, 8–13. (Russian. Uzbek summary)

The problem is

$$Lu\equiv u_{tt}-a^2u_{xx}=F(\mu,x,t,u,u_t,u_x)\ (0\leq x\leq 1,\ 0\leq t\leq 1)$$
$$u(0,t)=0=u(1,t),\ u(x,0)=u(x,1),\ u_t(x,0)=u_t(x,1).$$

Essential assumptions: $a$ is an odd number; $F$ is odd in $x$ and satisfies the boundary and periodicity conditions in $x$ and $t$, and $F_x$, $F_t$, $F_u$, $F_{u_t}$, and $F_{u_x}$ are Lipschitz continuous in all their arguments together (except the parameter) for $|u|\leq A$, $|u_t|\leq A$, and $|u_x|\leq A$ (whence $F$ is then also Lipschitz continuous);

$$f(\mu,x,t)\equiv F(\mu,x,t,0,0,0)=\sum_{n=1}^{\infty}c_{2n+1}(t)\sin(2n+1)\pi x,$$

where the $c$'s are Fourier coefficients. Conclusion: if $|a|>3M/2$, where $M$ is the maximum of the Lipschitz constants, then (without restriction on $\mu$) the problem has a unique solution. Starting with the solution $u_0$ of the problem with right member $f(\mu,x,t)$, for $k=0,1,\cdots$, $u_{k+1}$ satisfies the equation with right member $F(\mu,x,t,u_k,u_{kt},u_{kx})$; convergence is established from explicit expressions for the $u_k$ and an appeal to Arzela's theorem. *F. A. Ficken* (Knoxville, Tenn.).

**Sokolov, G. T. On periodic solutions of a class of partial differential equations.** Dokl. Akad. Nauk Uzbek. SSR. **1953**, no. 12, 3–7. (Russian. Uzbek summary)

The problem is

$$Lz\equiv z_{tt}-a^2z_{xx}=\Phi(x,t)+\mu f(z)\ (0\leq x\leq 1,\ 0\leq t\leq 1)$$
$$z(0,t)=0=z(1,t),\ z(x,0)=z(x,1),\ z_t(z,0)=x_t(x,1).$$

Essential assumptions: $\Phi(x,t+1)=\Phi(x,t)$, $\Phi(x,t+\frac{1}{2})=-\Phi(x,t)$, and $\Phi_t(x,t)$ is continuous, and of bounded variation in $x$; $f$ is odd and smooth. Conclusion: if $a=4m/p$, where $m$ and $p$ are integers with $p$ odd, and $|\mu|<\pi|a|/NA$, where $N=$ sup $|f'(z)|$ and $A$ is a certain constant, then the problem has a unique solution. The method of approximation is similar to those of the two papers reviewed above; here the explicit formulas involve expansions in sines or cosines of $2(2n+1)\pi t$. All three papers refer to work by N. A. Artem'ev [Izv. Akad. Nauk SSSR. Ser. Mat. **1937**, 15–50] and P. V. Solov'ev [ibid. **1939**, 149–164]. *F. A. Ficken* (Knoxville, Tenn.).

**Copson, E. T. On a regular Cauchy problem for the Euler-Poisson-Darboux equation.** Proc. Roy. Soc. London. Ser. A. **235** (1956), 560–572.

A new method is given for solving the wave equation in space of any odd number of dimensions without making use of any device for evaluating divergent integrals. This gives the clue to a simple method of solving the regular Cauchy problem for the so-called Euler-Poisson-Darboux equation in space of any odd number of dimensions, a problem which has previously only been solved by a very difficult extension of Marcel Riesz's method. (Author's summary.) *A. E. Heins.*

**Collatz, L. Fehlerabschätzungen für Näherungslösungen parabolischer Differentialgleichungen.** An. Acad. Brasil. Ci. **28** (1956), 1–9.

Extending to $n$ dimensions results by H. Westphal [Math. Z. **51** (1949), 690–695; MR **11**, 252], the author obtains a maximum principle, uniqueness theorems and error estimates for the solutions of differential equations of the form $\partial u/\partial t=g(u)-h(x_j,t,\partial u/\partial x_j,\partial^2u/\partial x_j^2)$ and similar equations. Here $g'(u)\leq 0$ and $h$ is monotone non decreasing in the $\partial^2u/\partial x_j^2$ at fixed $x_j$, $t$, $\partial u/\partial x_j$. The domain considered is cylindrical in $t$ and the boundary conditions may be non-linear, but of special type. At $t=0$, the boundary condition reduces to $u=0$. As an application of the results the following two boundary problems are solved numerically: $u_t=1+u_x$ for $0<x<1$, $t>0$ with $u=0$ for $t=0$; $u=0$ for $x=0$, $t>0$; $\partial u/\partial x=0$ for $x=1$, $t>0$. Secondly, $u_t=1+u_{xx}+u_{yy}$ for $|y|<1$, $|x|<1$, $t>0$ with $u=0$ for $t=0$; $u$ equal its normal derivative for $|x|=1$, $t>0$ and $|y|=1$, $t>0$. A linear combination of solutions of each equation is formed to give increasingly better approximations to the boundary conditions. The estimate of the error on the boundary gives an estimate of the error inside the domain. There are several minor misprints. *M. Steinberg* (Los Angeles, Calif.).

**Ciliberto, Carlo. Sulle equazioni quasi-lineari di tipo parabolico in due variabili.** Ricerche Mat. **5** (1956), 97–125.

Let $T$ denote the rectangular shaped domain $0<x<X$, $0<y<Y$; and set up the boundary value problem

$$(*)\qquad u_{xx}+au_y+bu_x+cu=g\ ((x,y)\in T),$$
$$(**)\qquad u(x,0)=u(0,y)=u(X,y)=0\ (0\leq x\leq X;\ 0\leq y\leq Y).$$

Here $a$, $b$, $c$, $g$ are functions of $x$, $y$, $u$, $u_x$ and $a<0$. The continuity conditions placed on the functions $a$, $b$, $c$, $g$ are too detailed to give here. However, roughly speaking, if the first partial derivatives of $a$, $b$, $c$, $g$ satisfy Hölder conditions the author proves the existence and uniqueness of a function $u=u(x,y)$ satisfying (*) and (**) in case $a_u\equiv 0$ or in case $a_{u_x}\neq 0$. [For earlier results of the author on this problem see Ricerche Mat. 3 (1954), 129–165; MR **16**, 1028.] *F. G. Dressel* (Durham, N.C.).

**Douglas, Jim, Jr. On the numerical integration of quasi-linear parabolic differential equations.** Pacific J. Math. **6** (1956), 35–42.

The differential equation

$$(1)\qquad \frac{\partial^2u}{\partial x^2}=F(x,t,u)\frac{\partial u}{\partial t}+G(x,t,u)\ (F\geq m>0),$$

in the region $0 \leq x \leq 1$, $t \geq 0$, is replaced by the following implicit difference equation: Put $u_{ij} = u(i\Delta x, j\Delta t)$, use the customary second difference approximation for the second derivative of $u_{ij}$ but use the backward difference $(u_{ij} - u_{i,j-1})(\Delta t)^{-1}$ for the time derivative and evaluate $F$ and $G$ at $u_{i,j-1}$. The author proves that if (1) has a solution $u(x, t)$ with bounded fourth derivatives then the solution of the difference equation converges to $u(x, t)$. He also shows that if $\Delta x = \lambda(\Delta t)^\alpha$ where $\lambda$ is fixed, then the most efficient computation scheme will be that for which $\alpha = \frac{1}{2}$. *B. Friedman* (New York, N.Y.).

**Pogorzelski, W. Etude de la solution fondamentale de l'équation parabolique.** Bull. Acad. Polon. Sci. Cl. III. 4 (1956), 9–13.

Let $a_{\alpha\beta}$, $b_\alpha$, $c$ be continuous functions of the $n+1$ variables $(x_1, \cdots, x_n; t)$ in the bounded and measurable region $\Omega$ in a Euclidean space of $n$ dimensions and for $t$ in the interval $0 \leq t \leq T$. The quadratic form $\sum a_{\alpha\beta}\lambda^\alpha\lambda^\beta$ is assumed to be positive definite on the product region $(\Omega, (0, T))$. The main objective of the present paper is to prove the existence of a fundamental solution (a function playing the same role that $(y-\eta)^{-\frac{1}{2}}\exp(-(x-\xi)^2/4(y-\eta))$ does in connection with $u_{xx} - u_y = 0$) for the parabolic equation

$$(*) \qquad \sum_{\alpha,\beta=1}^{n} a_{\alpha\beta}\frac{\partial^2 u}{\partial x_\alpha \partial x_\beta} + \sum_{\alpha=1}^{n} b_\alpha \frac{\partial u}{\partial x_\alpha} + cu - \frac{\partial u}{\partial t} = 0,$$

under the assumption that $a_{\alpha\beta}$, $b_\alpha$, $c$ satisfy a Hölder condition of order $h$ $(0 < h \leq 1)$ relative to the variables $x_1, \cdots, x_n$. Earlier proofs of the existence of a fundamental solution of (*), in general, assumed certain differentiability and Hölder conditions on the coefcients $a_{\alpha\beta}$ [Dressel, Duke Math. J. 7 (1940), 186–203; MR 2, 204; Giraud, C. R. Acad. Sci. Paris **195** (1932), 98–100]. The present note contains an outline of the author's attack on the above problem. The proofs are in the paper reviewed below. *F. G. Dressel* (Durham, N.C.).

**Pogorzelski, W. Étude de la solution fondamentale de l'équation parabolique.** Ricerche Mat. 5 (1956), 25–57.

In this paper the author proves the theorems he announced in the paper reviewed above. *F. G. Dressel.*

**Krzyżański, M. Sur l'allure asymptotique des solutions d'équation du type parabolique.** Bull. Acad. Polon. Sci. Cl. III. 4 (1956), 247–251.

Assume that the coefficients $a(x, t) > 0$, $b(x, t)$, $c(x, t) \leq 0$ of the parabolic operator $L(u) \equiv a(x, t)u_{xx} + b(x, t)u_x + c(x, t)u - u_t$ are continuous functions interior to $\Sigma$: $0 \leq x \leq 1$, $0 \leq t$; and, following the author, call a function regular in $\Sigma$ if it is continuous there and has second derivatives which are continuous interior to $\Sigma$. We state only the main result of the paper. If there exists a positive function $V(x, t)$ which is regular in $\Sigma$ with $\lim_{t\to\infty} V(x,t) = 0$ holding uniformly with respect to $x$ in $0 \leq x \leq 1$ and such that $L(V) \leq 0$ interior to $\Sigma$, then every solution $u$ of $L(u) = 0$ which is regular in $\Sigma$ and for which

$$\lim_{t\to\infty} u(0, t) = \lim_{t\to\infty} u(1, t) = 0$$

has the property that the following limit holds uniformly with respect to $x$ in $0 \leq x \leq 1$

$$\lim_{t\to\infty} u(x, t) = 0.$$

*F. G. Dressel* (Durham, N.C.).

**Rektorys, Karel. Two theorems concerning the equation $\partial u/\partial t = \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2$.** Časopis Pěst. Mat. 79 (1954), 333–366. (Czech. Russian and English summaries)

The author proves the following two theorems: Theorem I: Let $f(x)$ be a bounded function on $\langle 0, \pi \rangle$ and continuous almost everywhere. Then there exists one and only one function $V(x, y)$ in the rectangle $0 \leq x \leq \pi$, $0 \leq y \leq \pi$ satisfying the following conditions. (a) $V$ is harmonic in the interior of the rectangle. (b) $V$ has the boundary values $V(x, 0) = f(x)$ and otherwise zero. (c) $V$ is bounded. (d) $V$ is continuous as function of $x$ and $y$ everywhere where the boundary function is continuous. Theorem II: Let $f(x, y)$ be a bounded function in $0 \leq x \leq \pi$, $0 \leq y \leq \pi$ and continuous almost everywhere. Then there exists one and only one function $V(x, y, t)$ in $D$: $0 \leq x \leq \pi$, $0 \leq y \leq \pi$, $t \geq 0$ satisfying the following conditions. (a) $V$ satisfies in the interior of $D$ the heat equation $\partial V/\partial t = \Delta V$. (b) $V$ has the boundary values $V(x, y, 0) = f(x, y)$ and otherwise zero. (c) $V$ is bounded. (d) $V$ is continuous at every continuity point of the boundary function. The proof of these theorems which may be considered as well-known is obtained by analysis of the series for the solutions constructed formally by separation of variables. *C. Loewner* (Stanford, Calif.).

**Ahundov, A. M. On the theory of a problem with initial conditions for a system of partial differential equations in a real domain.** Akad. Nauk Azerbaĭdžan. SSR. Trudy Inst. Fiz. Mat. 6 (1953), 62–87. (Russian. Azerbaijani summary)

The author considers the system

$$\frac{\partial^{n_i m} U_i}{\partial t_1^{n_i} \partial t_2^{n_i} \cdots \partial t_m^{n_i}} = \sum_{j=1}^{N} \sum_{(k_s)} A_{ij}^{(k_0, k_1, \cdots, k_n)}(\bar{t}) \frac{\partial^{k_0 m + k_1 + \cdots + k_n} U_j}{\partial t_1^{k_0} \cdots \partial t_m^{k_0} \partial x_1^{k_1} \cdots \partial x_n^{k_n}} + f_i(\bar{t}, \bar{x})$$

with the initial conditions

$$\left.\frac{\partial^k U_i}{\partial t^k_V}\right|_{t_V = t_V^0} = \phi_i^{(k)}(x_1, \cdots, x_n),$$

where $i = 1, \cdots, N$; $V = 1, \cdots, m$; and $k = 0, 1, \cdots, (n_i - 1)$, for the region $0 \leq t_s \leq T_s$, $-\infty < x_k < +\infty$. The notation $\sum_{(k_s)}$ means a summation over all terms where $k_j \geq 0$, $k_0 m + k_1 + \cdots + k_n \leq K$ for some $K$; it is also assumed that in the coefficients $A_{ij}^{(k_0, k_1, \cdots, k_n)}(\bar{t})$ we have $k_0 < n$ for all $i$ and for all combinations $k_1, \cdots, k_n$; that is to say, in the $j$th equation the highest derivative of $U_j$ with respect to the $t$'s occurs on the left hand side of the equation. For this system the author proves that the solutions $U_i$ exist, are uniquely determined, and depend in a continuous way on the initial data. The proof uses Fourier transforms with respect to $\bar{x}$ to reduce to a system of the form

$$\frac{\partial^m v_j}{\partial t_1 \cdots \partial t_m} = \sum_{j=1}^{N} \sum_{(k_s)} A_{ij}^{(k_1, \cdots, k_n)}(\bar{t})(i\alpha_1)^{k_1} \cdots (i\alpha_n)^{k_n} v_j,$$

$$\left. v_i \right|_{t_s = t_s^0} = \phi_i(\alpha_1, \cdots, \alpha_n).$$

The method is similar to one used by Š. A. Iskenderov [same Trudy 4–5 (1952), 106–127; MR 17, 860] who dealt with the case $m = 1$. *R. B. Davis* (Syracuse, N.Y.).

**Garnir, H. G.; et Thyssen, M. Solution du problème de Cauchy pour quelques opérateurs de la physique mathématique.** Bull. Soc. Roy. Sci. Liège 25 (1956), 27–49.

On donne les opérateurs différentiels hyperboliques

(A) $\partial^2/\partial t^2 - c^2 \sum_{i=1}^{n} \partial^2/\partial x_i^2 - k$,

(B) $(\partial^2/\partial t^2 - \sum_{i=1}^{n} \partial^2/\partial x_i^2 - k)(\partial^2/\partial t^2 - \sum_{i=1}^{n} a_i^2 \partial^2/\partial x_i^2 - l)$;

les A. s'intéressent aux cas $n=1, 2, 3$. La solution élémentaire de (A) (resp. (B)) à support dans un demi cône situé dans $t \geqq 0$ est connue [cf. H. G. Garnir, même Bull. 20 (1951), 174–194, 271–296; MR 13, 243, 352; M. Thyssen, Archives originales du Centre Nat. Rech. Sci., no. 353 (1954), 1–30]. A l'aide de la solution élémentaire, la solution du problème de Cauchy s'exprime par un produit de composition; les A. étudient la régularité de ce produit de composition, selon la régularité des données initiales et du second membre, et donnent de nombreuses formules explicites, trop longues pour être rapportées ici.

*J. L. Lions.*

**Kislicyn, S. G. On approximate solution of some problems of mathematical physics.** Leningrad. Gos. Ped. Inst. Uč. Zap. 89 (1953), 121–137. (Russian)

This is an expository article showing how the partial differential equations of mathematical physics are replaced by difference equations utilizing functional values at nodes of a square net. A number of drawings illustrate various boundary problems. Application is made to the deformation of an infinite rubber strip subjected to pressure on two parallel surfaces, and to the solution of problems of elasticity by use of complex variables.

*W. E. Milne* (Corvallis, Ore.).

**See also:** Stoker, p. 42; Lax and Richtmyer, p. 48; Doob, p. 76; Hunt, p. 77; Wintner, p. 66; Durand, p. 72; Bergman, p. 73; Jefferies, p. 102; Fleishman, p. 88; Rahmatulin, p. 88; Barenblatt, p. 91.

## *Difference Equations, Functional Equations*

**Lax, P. D.; and Richtmyer, R. D. Survey of the stability of linear finite difference equations.** Comm. Pure Appl. Math. 9 (1956), 267–293.

In dieser Arbeit wird die Konvergenz der Lösungen approximierender Differenzengleichungen gegen die Lösungen von Anfangswertproblemen bei festem $t$ und kleiner werdender Maschenweite zunächst durch allgemeine Betrachtungen in Banachräumen untersucht. Diese „Stabilität" (der Einfluß von Aufrundungsfehlern wird nicht berücksichtigt) wird für sinnvoll gestellte (properly posed) Anfangswertprobleme als gleichwertig mit der Approximation des erzeugenden Operators der Halbgruppe, die zu dem gegebenen Problem gehört, und einer Beschränktheitseigenschaft der Lösungen der Differenzengleichungen erkannt. Daraus ergeben sich bei Anwendung auf lineare partielle Differentialgleichungen mit konstanten Koeffizienten, die mittels Fourierreihen behandelt werden, gewisse Bedingungen von J. v. Neumann und K. Friedrichs. Spezielle Beispiele (Wellengleichung und Diffusionsgleichung) erläutern die allgemeine Theorie.

*D. Morgenstern* (Berlin).

**Cunningham, W. J. Nonlinear oscillators with constant time delay.** J. Franklin Inst. 261 (1956), 495–507.

The author, using averaging techniques, discusses the equation

$$x'(t)+ax(t)-cx(t-\tau)+g[ax^3(t)+3x^2(t)x'(t)]=0,$$

which arises as indicated in the title. *R. Bellman.*

**Hosszú, Miklós. On analytic half-groups of complex numbers.** Publ. Math. Debrecen 4 (1956), 459–464.

Let $F(x, y)$ be a binary operation from a connected domain $D$ of complex numbers into $D$ $(x, y, F \in D)$. $D$ is called an "analytic half-group" with operation $F$ if $F$ is differentiable on $D$ and satisfies there the functional equation $F[F(x, y), z]=F[x, F(y, z)]$. A. Kuwagaki [Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. 27 (1953), 225–234; MR 15, 324] completely determined the character of $F$ when $F(0, 0)=0$. The author lifts this last restriction and obtains a corresponding characterization of the function $F(x, y)$. *I. M. Sheffer* (University Park, Pa.).

**See also:** Sade, p. 3; Bochner and Chandrasekharan, p. 19; Douglas, p. 46; Pál, p. 76; Byrd, p. 90.

## *Integral and Integrodifferential Equations*

**Aumann, Georg. Zur Existenz eines Eigenwertes einer Integralgleichung.** Math. Nachr. 14 (1955), 73–74.

The author gives a new proof of the existence of eigenvalues for a non-zero continuous real symmetric kernel; the proof rests on a variational argument.

{Reviewer's note. A somewhat simpler proof, on similar lines, can be constructed by the method used in the reviewer's paper in Bull. Amer. Math. Soc. 44 (1938), 835–836.} *F. Smithies* (Cambridge, England).

**Gagaev, B. M. On the existence of eigenvalues of integral equations whose kernels are entire rational functions of a parameter.** Ukrain. Mat. Ž. 4 (1952), 120–123. (Russian)

Let $L(x, y, \lambda)=\sum_{i=0}^{m} \lambda^i G_i(x, y)$ be a bounded kernel satisfying the usual continuity restrictions of the Fredholm theory, and let $R(x, y, \lambda)$ be its resolvent kernel. The author proves the following results.

Suppose that the Fredholm determinant $D(\lambda)$ of $L(x, y, \lambda)$ is not identically zero. Then (i) a necessary and sufficient condition for $L(x, y, \lambda)$ to possess no eigenvalues is that

$$\left[\frac{d^n}{d\lambda^n}\int_a^b R(x, x, \lambda)dx\right]_{\lambda=0}=0 \quad (n\geqq 2m+1);$$

(ii) if (a) $G_i(x, y)>0$ $(0\leqq i\leqq m)$ or (b) $G_0(x, y)$ and $G_1(x, y)$ are symmetric, $G_0$ being positive definite, then $L(x, y, \lambda)$ has at least one eigenvalue. *F. Smithies.*

**Kasimov, D. M. On an integral equation whose kernel is an analytic function of a parameter.** Azerbaĭdžan. Gos. Univ. Trudy. Ser. Fiz.-Mat. 4 (1954), 47–60. (Russian. Azerbaijani summary)

The equation is

$$\varphi(x, \lambda)=f(x)+\int_a^b K(x, y; \lambda)\varphi(y, \lambda)\, dy,$$

where $f$ is continuous and for each $(x, y)$ in the square

$a \leq x, y \leq b$, $K(x, y; \lambda)$ is analytic in $\lambda$ in some region $B$ of the $\lambda$-plane. If $\lambda_0 \in B$ and

$$K(x, y, \lambda) = \sum_0^\infty K_n(x, y)(\lambda-\lambda_0)^n,$$

where $|K_n(x, y)| \leq \mu$ (const) and 1 is not a characteristic value of $K_0(x, y)$, then $K$ is said to be nonsingular at $\lambda=\lambda_0$. In that case, as is shown by successive approximation, there exists a unique solution $\varphi(x, \lambda)$ which is analytic in a neighborhood of $\lambda=\lambda_0$. *F. A. Ficken.*

**Pöschl, Klaus. Über eine spezielle Integralgleichung.** Z. Angew. Math. Mech. 36 (1956), 161–167. (English, French and Russian summaries)

The integral equation in question, encountered in evaluating the tension in the wires of vacuum tube grids, is

$$(1) \quad u(t, s)-\lambda \int_0^s K(t, \eta)u(\eta, s)d\eta = f(t, s) \quad (0 \leq s \leq 1, 0 \leq t \leq 1),$$

which for fixed $s$ is an ordinary Fredholm integral equation of the second kind.

The resolvent kernel $\Gamma(t, \eta, s, \lambda)$, a function of four variables, allows the solution of (1) to be written as

$$(2) \qquad u(t, s) = f(t, s)+\lambda \int_0^s \Gamma(t, \eta, s, \lambda)f(\eta, s)d\eta$$

for

$$|\lambda| \leq \left[\int_0^1 \int_0^1 K^2(t, \eta)d\eta dt\right]^{-\frac{1}{2}}.$$

By a simple argument it is established that (1) has no eigenvalues, although $\Gamma$, for fixed $s$, has poles at the eigenvalues $\lambda=\lambda_k(s)$ of

$$(3) \qquad \varphi(t) = \lambda \int_0^s K(t, \eta)\varphi(\eta)d\eta.$$

For the kernel $K(t, \eta)$ continuous and symmetric on the unit square, it is shown that $|\lambda_k(s)|$ is continuous and monotone non-increasing. If the eigenvalue $\lambda_k(s)$ is simple and corresponding eigenfunctions at $s$ and at $s'$ are denoted by $\varphi_s(t)$ and $\varphi_{s'}(t)$, then the rate of change of $\lambda_k(s)$ with respect to $s$ is given by

$$(4) \qquad \frac{1}{\lambda_k(s)} \frac{d\lambda_k(s)}{ds} = -\frac{\varphi_{1s}{}^2(s)}{\int_0^s \varphi_{1s}{}^2(t)dt},$$

where $\varphi_{1s}(t) = \lim_{s' \to s} \varphi_{s'}(t)$, uniformly with respect to $t$ on $0 \leq t \leq s'$. If the eigenvalue $\lambda_k(s)$ is $r$-fold, then a by-product of the proof leading to (4) shows that at least $r-1$ corresponding eigenfunctions $\varphi_{\mu s}(t)$ $(\mu=1, 2, \cdots, r)$, vanish at $0=s$.

The paper concludes by showing that for fixed $s$ and $\lambda \neq \lambda_k(s)$, the resolvent kernel $\Gamma(t, \sigma, s, \lambda)$ is a continuous function of $t$ and $\sigma$ in the unit square and possesses a derivative with respect to $\lambda$. Moreover, $d\Gamma/ds$ exists and is continuous for $0 \leq s \leq 1$ and $\lambda \neq \lambda_k(s)$. *J. F. Heyda.*

**Busbridge, I. W. On solutions of the non-homogeneous form of Milne's first integral equation.** Quart. J. Math. Oxford Ser. (2) 6 (1955), 218–231.

Consider the integral equation

$$J(\tau) = \tfrac{1}{2}\chi \int_0^\infty E_1(|t-\tau|)J(t)dt + B(\tau)$$
$$= \chi\Lambda_\tau\{J(t)\} + B(\tau)$$

where $0<\chi<1$ is a constant and $E_n(\tau)$ is the $n$th exponential integral. Further let

$$j(s) = \mathfrak{L}_s\{J(t)\} \equiv s \int_0^\infty J(t)e^{-st}\, dt \quad (\operatorname{Re} s>0).$$

The following theorems are proved: Consider first the homogeneous case when $B=0$. Then theorem 1A: If $0<\chi<1$ and if $J(\tau)$ is continuous for $\tau>0$ and is such that

$$J(\tau) = O(\ln \tau^{-1}) \text{ as } \tau \to 0 \text{ and } J(\tau) = o(e^{\theta\tau}) \text{ as } \tau \to \infty,$$

for every $\theta>0$, then $J(\tau)=0$.

Consider next the case

$$B(\tau) = e^{-\sigma\tau} \ (\sigma \geq 0).$$

Then it is is shown that the solution is unique and that

$$j(\mu^{-1}, \sigma) = \mathfrak{L}_{1/\mu}\{J(t, \sigma)\} = \frac{H(\mu)H(\sigma^{-1})}{1+\mu\sigma},$$

where $H(\mu)$ is the unique solution of the integral equation

$$H(\mu) = 1+\tfrac{1}{2}\chi\mu H(\mu) \int_0^1 \frac{H(x)}{x+\mu}\, dx,$$

which is regular for $\operatorname{Re} \mu>0$. The essential part of the proof of this theorem consists in showing that

$$\frac{\partial}{\partial\tau} J(\tau,\sigma)+\sigma J(\tau,\sigma)-\tfrac{1}{2}\chi J(0,\sigma) \int_1^\infty J(\tau,x)\frac{dx}{x}$$
$$= \chi\Lambda_\tau\left\{\frac{\gamma t}{\delta} J(t, \sigma)+\sigma J(t, \sigma)-\tfrac{1}{2}\chi J(0, \sigma) \int_1^\infty J(t, x)\frac{dx}{x}\right\},$$

and that the quantity on the left-hand side satisfies the conditions of Theorem 1A and is therefore zero. Next, taking the Laplace transform of the same quantity the author shows after some manipulations that $j(\mu^{-1}, \sigma)$ can be expressed in the form required by the theorem stated.

The solution for the case when

$$B(\tau) = \sum_{\nu=0}^n \frac{a_\nu \tau^{n-\nu}}{(n-\nu)!}$$

is found. In this case, the "emergent intensity" $j(\mu^{-1})$ is found in the form

$$j(\mu^{-1}) = H(\mu)\{J_n(0)+\mu J_{n-1}(0)+\cdots+\mu^n J_0(0)\},$$

where the constants $J_\nu(0)$ are expressed in terms of the moments of the $H$-function. An explicit formula $J_n(0)$ is also given. Finally, the exact solution for the case $B(\tau)=E_n(\tau)$ is also found.

The case $\chi=1$ requires a separate treatment; the necessary modifications are stated. *S. Chandrasekhar.*

**Wiebelitz, R. Über den Zusammenhang zwischen Systemen linearer Differentialgleichungen und Volterrascher Integralgleichungen mit ausgeartetem Kern.** Arch. Math. 7 (1956), 184–196.

Under specified continuity conditions on the functions $X_{\alpha\beta}{}^j(s)$, $Y_{\alpha\beta}{}^j(s)$ and $f_\alpha(s)$, the solution of the system

$$(1) \qquad y_\alpha(s)+\sum_{\beta=1}^n \int_a^s k_{\alpha\beta}(s, t)y_\beta(t)dt = f_\alpha(s),$$

with $k_{\alpha\beta}(s, t) = \sum_{j=1}^{m_{\alpha\beta}} X_{\alpha\beta}{}^j(s) Y_{\alpha\beta}{}^j(t)$ $(\alpha, \beta=1, \cdots, n)$ is shown to be

$$(2) \quad y_\alpha(s) = f_\alpha(s)+\int_a^s \sum_{\alpha\beta=1}^n L_\alpha[u_{\alpha\beta}(s, t)]\cdot f_\beta(t)dt \quad (\alpha=1, \cdots, n),$$

the resolvent kernel being the $n$-rowed square matrix with elements

$$(3) \quad L_\alpha[u_{\alpha\beta}(s, t)] = \sum_{\nu=0}^{M_\alpha} a_{M_\alpha-\nu}{}^\alpha(s)\frac{\partial^\nu}{\partial s^\nu} u_{\alpha\beta}(s, t) \quad (\alpha, \beta=1, \cdots n),$$

where the $a_{M_i-\nu}{}^i(s)$ appear as the coefficients in the adjoint

differential equation

$L_i[u(s)]=$

$$\sum_{\gamma=0}^{M_i}(-1)^\gamma\frac{d^\gamma}{ds^\gamma}\{\varphi^i{}_{M_i-\gamma}(s)\cdot u(s)\}=\sum_{\gamma=0}^{M_i}a^i_{M_i-\gamma}(s)\frac{d^\gamma}{ds^\gamma}u(s)$$

associated with the differential equation

(4) $L_i^*[v(s)]\equiv\sum_{\lambda=0}^{M_i}\varphi^i_{M_i-\lambda}(s)v^{(\gamma)}(s)=0\ (i=1,\cdots,n),$

whose solutions are $Y_{\alpha i}{}^j$ $(\alpha=1,\cdots,n;j=1,\cdots,m_{\alpha i})$. The functions $u_{\alpha\beta}(s,t)$ appearing in (3) satisfy the system of differential equations

$$L_\alpha[u_\alpha(s)]+\sum_{i=1}^{n}\sum_{j=1}^{m_{\alpha i}}X_{\alpha i}{}^j(s)P_{\alpha i}{}^j[u_i(s)]=0\ (\alpha=1,\cdots,n),$$

the differential operators $P_{\alpha i}{}^j$ being determined from (applying Lagrange's identity to (4))

$$Y_{\alpha i}{}^j(s)L_i[u(s)]=\frac{d}{ds}P_{\alpha i}{}^j[u(s)];$$

and also satisfy the initial conditions

$$u_{\alpha\beta}=\frac{\partial}{\partial_s}u_{\alpha\beta}=\cdots=\frac{\partial^{M_i-2}}{\partial s^{M_i-2}}u_{\alpha\beta}=0;$$

$a_0{}^i\partial^{M_i-1}u_{\alpha\beta}/\partial s^{M_i-1}=\delta_{\alpha\beta}$ at the points $s=t$. (Here, $M_i\leq\sum_{\alpha=1}^{n}m_{\alpha i}$ and $a_0{}^i(s)\neq0$ over its interval of definition.)

A companion theorem considers the system of differential equations

(5) $a_i(x)y_i{}^{(m_{ii}+1)}(x)+\sum_{j=1}^{n}F_{ij}[y_j(x)]=$

$$a_i(x)\frac{d^{m_{ii}+1}y_i}{dx^{m_{ii}+1}}+\sum_{j=1}^{n}\sum_{\gamma=0}^{m_{ij}}a_{m_{ij}}{}^{ij}{}_{-\gamma}(x)\frac{d^\gamma y_j}{dx^\gamma}=f_i(x),$$

$$(i=1,\cdots,n).$$

It is shown that by determining the solution of the homogeneous subsystem of (5),

(6) $a_i(x)u_i{}^{(m_{ii}+1)}(x)+\sum_{j=1}^{m}F_{ij}[u_j(x)]=0\ (i=1,\cdots,m)$

the solution of (5) can be made to depend upon the solution of a system of $(n-m)$ linear Volterra integral equations of the third kind. Since the latter system can, by the first theorem, be solved by finding the solution of a system of $(n-m)$ linear differential equations, a method is at hand for solving the general system (5) in terms of the solution of a homogeneous subsystem of (5).

Detailed examples are given illustrating both theorems.

*J. F. Heyda* (Indianapolis, Ind.).

**Tricomi, Francesco G. Equazioni integrali singolari del tipo di Carleman.** Ann. Mat. Pura Appl. (4) **39** (1955), 229–244.

The author considers the integral equation

(1) $$a(x)\varphi(x)-\lambda\int_{-1}^{1}\frac{\varphi(y)}{y-x}dy=f(x),$$

where $a(x)$ is continuous in $(-1,1)$ and satisfies a Lipschitz condition of positive order near the points $\pm1$, the integral is taken as a principal value and equality is interpreted as holding almost everywhere. Write

$\alpha=\pi^{-1}\arctan[\lambda\pi/a(-1)]$, $\beta=\pi^{-1}\arctan[\lambda\pi/a(1)]$,

$$\gamma=\max(\alpha,1-\beta),$$

and suppose that $f(x)$ belongs to the class $L^q(-1,1)$, where $q(1-\gamma)>1$. The author shows that under these hypotheses, (1) has the solution

$\varphi(x)=$

$$\frac{a(x)f(x)}{a^2(x)+\lambda^2\pi^2}+\frac{\lambda e^{\tau(x)}(1-x)^{-1}}{\sqrt{[a^2(x)+\lambda^2\pi^2]}}\int_{-1}^{1}\frac{e^{-\tau(x)}(1-y)f(y)}{\sqrt{[a^2(y)+\lambda^2\pi^2]}}\frac{dy}{y-x}+\frac{Ce^{\tau(x)}(1-x)^{-1}}{\sqrt{[a^2(x)+\lambda^2\pi^2]}},$$

where $C$ is an arbitrary constant and

$$\tau(x)=\frac{1}{\pi}\int_{-1}^{1}\frac{\theta(y)}{y-x}dy,\ \theta(x)=\arctan\frac{\lambda\pi}{a(x)}.$$

This solution belongs to the class $L^r$ for $1<r<q(1+\gamma q)^{-1}$, and the term containing the constant $C$ is the most general solution of the associated homogeneous equation belonging to any class $L^p(-1,1)$ for $p>1$.

If $q(1-\beta)>1$, the solution can also be written in Carleman's original form [Ark. Mat. Astr. Fys. **16** (1922), no. 26 , in which the factors $(1-x)^{-1}$ and $(1-y)$ are omitted from the second term.

{Reviewer's note. The author does not refer to the work of the Tbilisi school on equations of this type [see, e.g., N. I. Mushkelishvili, Singular integral equations, Gostehizdat, Moscow-Leningrad 1946; MR **11**, 523; **15**, 434 .}

*F. Smithies* (Cambridge, England).

**Lehner, Joseph; and Wing, G. Milton. Solution of the linearized Boltzmann transport equation for the slab geometry.** Duke Math. J. **23** (1956), 125–142.

The authors consider an initial value problem for the equation

$$\frac{\partial u}{\partial t}=Au\equiv-\mu\frac{\partial u}{\partial x}+\frac{c}{2}\int_{-1}^{1}u(x,\mu',t)\,d\mu',$$

where $u(x,\mu,t)$ is defined for $|x|\leq a$, $|\mu|\leq1$, $t>0$, and satisfies the conditions

$u(a,\mu,t)=0\ (\mu<0)$, $u(-a,\mu,t)=0\ (\mu>0)$,

$$u(x,\mu,+0)=f(x,\mu).$$

In an earlier paper [Comm. Pure Appl. Math. **8** (1955), 217–234; MR **16**, 1120], the authors showed that the point spectrum of $A$, regarded as an operator in the appropriate Hilbert space, is a finite non-empty set of positive real numbers $\beta_1,\cdots,\beta_m$ and that its continuous spectrum is the half-plane $\Re(\lambda)\leq0$. Denoting by $\psi_j$ and $\psi_j^*$ the normalised eigenfunctions of $A$ and its adjoint $A^*$ for the eigenvalue $\beta_j$, and taking $0<\gamma<\min\beta_j$, the authors now show that the solution of the initial value problem can be expressed, by using semigroup theory, in the form

$$u=\sum_{j=1}^{m}(f,\psi_j^*)\psi_je^{\beta_jt}+\lim_{\omega\to\infty}\frac{1}{2\pi i}\int_{\gamma-i\omega}^{\gamma+i\omega}e^{\lambda t}R(\lambda,A)fd\lambda$$

$$=\sum_{j=1}^{m}(f,\psi_j^*)\psi_je^{\beta_jt}+\zeta(x,\mu,t),$$

where $R(\lambda,A)$ is the resolvent of $A$. Under additional assumptions (mainly of smoothness) on the function $f(x,\mu)$, it is shown that $\zeta(x,\mu,t)\to0$ as $t\to\infty$ for almost all $(x,\mu)$; thus for large $t$ the solution $u(x,\mu,t)$ can be expressed with good approximation in terms of the eigenfunctions. Finally, it is shown that if $f$ is orthogonal to all the $\psi_j^*$, then the convergence of $\zeta(x,\mu,t)$ to 0 as $t\to\infty$ is not exponentially rapid.

*F. Smithies.*

**Mikusiński, J. Sur quelques équations intégro-différentielles.** Studia Math. 15 (1956), 182–187.

The author proves four uniqueness theorems for three homogeneous integro-differential equations. Two of the equations are

$$\int_0^t p(t-\tau)x_\lambda(\lambda, \tau)d\tau = \int_0^t q(t-\tau)x(\lambda, \tau)d\tau,$$

$$\int_0^t p(t-\tau)x_\lambda(\lambda, \tau)d\tau = x_\lambda(\lambda, t),$$

and the third equation is like the second but with $x$ and $x_\lambda$ exchanged in position. The function $x(\lambda, t)$ and its derivative $x_\lambda$ are assumed continuous in a rectangle $0 \leq t \leq T$, $0 \leq \lambda \leq \Lambda$, with $x(0, t) \equiv 0$. The functions $p$ and $q$ are assumed integrable on $[0, T]$. The second equation then implies $x(\lambda, t) \equiv 0$ in the rectangle. Likewise for the third equation if $p(t) \leq 0$, and for the first or third equation $p(t) \geq 0$ and $\int_0^t p(\tau)d\tau \geq kt^\alpha$, where $k$ and $\alpha$ are constants, with if $k>0$, $0<\alpha<1$. Finally, if $p$ and $q$ are absolutely continuous and $p(0) \neq 0$, the first equation implies $x(\lambda, t) \equiv 0$.

The methods are entirely different from those used by the author and others in dealing with similar problems for a triangular region instead of the present rectangular one Finkelsztejn, Mikusiński, and Ryll-Nardzewski [Colloq. Math. 2 (1951, 178–181; MR 14, 54]. *A. E. Taylor.*

**See also:** Liboff, p. 30; Filin, p. 43; Albert, p. 72; Mertens, p. 102; Chow, p. 87; Nowacki, p. 82; Dol'berg, p. 84; Marziani, p. 94; Mayer, p. 82.

### *Calculus of Variations*

**Bertolini, Fernando. Su un problema di calcolo delle variazioni studiato da M. R. Gibrat.** Rend. Circ. Mat. Palermo (2) 5 (1956), 43–58.

The author considers a maximization problem suggested by the question of regulating optimally the flow of sea water collected in a tidal basin through hydroelectric generators. The problem reduces to maximizing a regular variational integral $\int F(t, z, z')\,dt$, subject to inequality constraints of the form $h(t) \leq z \leq Z$ and $g(z, t) \leq z' \leq f(z, t)$. The paper is mainly devoted to proving the existence and differentiability of a solution $z_0(t)$. It leaves open such intriguing questions as whether, under appropriate further restrictions on $F$, $h$, $\cdots$, the solution may have some simple form (e.g., the set where each inequality becomes an equality is an interval, and the order of these intervals can be determined). In the paper reviewed below this question is answered affirmatively for a similar class of problems. *W. H. Fleming* (Lafayette, Ind.).

**Bellman, R.; Fleming, W. H.; and Widder, D. V. Variational problems with constraints.** Ann. Mat. Pura Appl. (4) 41 (1956), 301–323.

This paper considers the problem of maximizing an integral $\int_0^T k(x, x')dt$ subject to the differential inequalities $u(x) \leq x' \leq v(x)$, where $u$ and $v$ are given functions, and to the initial condition $x(0)=c$. (Two other forms of the problem are considered, which are equivalent to the preceding, under certain assumptions.) Existence theorems, necessary conditions, conditions for uniqueness of the solution, and conditions for differentiability of the maximizing function are obtained. Examples arising from the theory of dynamic programming are considered. A partial discussion is given of the $n$ dimensional case, in one of the alternative forms. *L. M. Graves.*

**Rodov, A. M. On the derivation of a general expression for the first variation.** Belorussk. Gos. Univ. Uč. Zap. Ser. Fiz.-Mat. 15 (1953), 22–25. (Russian)

A familiar derivation of Euler's variational equations, eliminating variations in favor of derivatives.

*A. W. Wundheiler* (Chicago, Ill.).

**See also:** Beckert, p. 45; Rosenbloom, p. 71.

## TOPOLOGICAL ALGEBRAIC STRUCTURES

### *Topological Groups*

**Preston, Gerald C. On locally compact totally disconnected Abelian groups and their character groups.** Pacific J. Math. 6 (1956), 121–134.

The Pontrjagin duality theorem is valid for all locally compact abelian groups only when the groups in question are continuous homomorphisms (characters) of each other into the circle $K$, the character group being given the compact-open topology. The author shows that for totally disconnected locally compact abelian groups, $K$ can be replaced by the rationals mod 1 under various topologies (including the discrete). The character groups are not necessarily identical with those of the Pontrjagin theory. *P. S. Mostert* (New Orleans, La.).

**Følner, Erling. On groups with full Banach mean value.** Math. Scand. 3 (1955), 243–254 (1956).

A bi-invariant Banach mean value on a group $G$ is a non-negative linear functional on the vector-lattice of all bounded real-valued functions on $G$ which is invariant under both right and left translations. The author proves that a necessary and sufficient condition for a group $G$ to have a bi-invariant Banach mean value is that there exists a number $k$, $0<k<1$, such that for every finite sequence $\{a_1, \cdots, a_n\}$ of elements of $G$ there exists a finite subset $E$ of $G$ such that $n^{-1}\sum_1^n N(E \cap Ea_i) \geq kN(E)$, where $N(\cdot)$ denotes the number of elements in the set between the parentheses. *L. H. Loomis.*

**See also:** Nobusawa, p. 7; Prékopa, Rényi and Urbanik, p. 25; Narumi, p. 95.

### *Lie Groups, Lie Algebras*

**Curtis, Charles W. Modular Lie algebras. I.** Trans. Amer. Math. Soc. 82 (1956), 160–179.

The title refers to a class of Lie algebras over fields of characteristic $p>0$ defined in terms of semisimple Lie algebras of characteristic 0 and retaining many properties of them. The construction is roughly as follows: In a semisimple Lie algebra $\mathfrak{L}$ over a field $C$ of characteristic 0 a canonical basis relative to a Cartan subalgebra is selected in such a way that the structural constants generate an algebraic number field $K$. Suppose $\mathfrak{p}$ is any one of a class of prime ideals in $K$ and $\mathfrak{o}$ the ring of $\mathfrak{p}$-integers in $K$. Consider the Lie subring $\Sigma$ of $\mathfrak{L}$ which is the $\mathfrak{o}$-submodule of $\mathfrak{L}$ generated by the basis selected. Now map $\Sigma$ into $\mathfrak{l}=\Sigma/p\Sigma$ by taking the coefficients of each element modulo $p$. $\mathfrak{l}$ is a modular Lie algebra over $\mathfrak{o}/\mathfrak{p}$. Modular Lie algebras have non-degenerate Killing forms.

The remainder of the paper is devoted to a demonstration that basic structural and representational properties of $\mathfrak{L}$ remain approximately correct at least in $\mathfrak{l}$. In particular Cartan subalgebras, roots, and weights of representations behave in $\mathfrak{l}$ very much as in $\mathfrak{L}$. *W. G. Lister.*

**Albert, A. A.; and Frank, M. S. Simple Lie algebras of characteristic $p$.** Univ. e Politec. Torino. Rend. Sem. Mat. **14** (1954–55), 117–139.

The authors reveal several new infinite classes of simple Lie algebras of characteristic $p>2$ which appear as subalgebras of the Witt algebras of derivations of certain polynomial algebras, and they also develop a general multiplication table method for constructing Lie algebras of characteristic $p$ out of $p$-groups, linear, and bilinear functions. The latter algebras or, in certain cases, homomorphic images or subalgebras of homomorphic images of them turn out to be simple.

The new simple derivation algebras, in addition to the Witt algebras and the class recently described by the second author [Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 713–719; MR **16**, 562], comprise two infinite classes of algebras with dimensions $(n-1)p^n$ and $p^{2n}-2$. The new simple algebras which are specified by basis products fall into three infinite classes of dimensions $p^n$, $p^n-1$, $p^n-2$.

*W. G. Lister* (Providence, R.I.).

**Ono, Takashi. On birational invariance of classical groups.** J. Math. Soc. Japan **8** (1956), 167–175.

Soient $V$ un espace vectoriel de dimension $n$ sur un corps commutatif $K$, $f$ une forme alternée (resp. quadratique) non dégénérée sur $V$, $\mathrm{Sp}(V, f)$ (resp $\mathrm{O}(V, f)$) le groupe symplectique (resp. orthogonal) correspondant. Soit $G'$ un groupe algébrique d'automorphismes de $V$ tel qu'il existe un isomorphisme birationnel de $\mathrm{Sp}(V, f)$ (resp. $\mathrm{O}(V, f)$ sur $G'$. Si $K$ est de caractéristique 0 et si $n>6$ et $n\neq 8$, l'auteur montre alors que l'on a nécessairement $G'=\mathrm{Sp}(V, g)$ (resp. $G'=\mathrm{O}(V, g)$). *J. Dieudonné.*

**See also:** Albert and Frank, p. 52.

## *Topological Vector Spaces*

**Heider, L. J. A characterization of function-lattices.** Duke Math. J. **23** (1956), 297–301.

Let $C^*(X, T)$ denote the set of all bounded real-valued functions which are defined on a set $X$ and are continuous under a given topology $T$. A real-valued function $\varphi$ defined on $C^*(X, T)$ preserving both lattice (ring) operations of $C^*(X, T)$ and satisfying the condition $\varphi(r)=r$ for all constant functions $r(x)\equiv r$ of $C^*(X, T)$ is called a real lattice homomorphism (real ring homomorphism). The definition of a real lattice homomorphism on a lattice which contains a sublattice lattice-isomorphic to the set of all real numbers is analogous. For each point $x$ in $X$, the relation $\varphi(f)=f(x)$ determines a real lattice homomorphism $\varphi$ on $C^*(X, T)$; such a real lattice homomorphism is called point real lattice homomorphism. Theorem 2.1: If $(X, T)$ is a compact Hausdorff space, then the real lattice homomorphism and real ring homomorphisms on $C^*(X, T)$ coincide, and are exactly the mappings $\varphi(f)=f(x)$ defined by the points $x$ of $(X, T)$.

Let $X$ be a set with the discrete topology, $\beta X$ its Stone-Čech compactification. Let $B(X)$ denote the collection of all bounded real-valued functions on $X$; then every $f$ in $B(X)$ has a continuous extension $\bar{f}$ to $\beta X$. Let $C^*(X)$ denote any sublattice of $B(X)$ with the following properties: (1) $C^*(X)$ includes the constant functions; (2) if $x\neq y$ in $X$, there is a function $f$ in $C^*(X)$ such that $f(x)\neq f(y)$; (3) for each point $\alpha$ in $\beta X$, there is a point $x$ in the set $X$ such that $\bar{f}(\alpha)=f(x)$ for all functions $f$ in $C^*(X)$. Theorem 4.1: Every sublattice of $B(X)$ with properties (1), (2), and (3) is contained in a sublattice of $B(X)$ maximal with respect to these properties. The latter sublattices of $B(X)$ are exactly the lattices $C^*(X, T)$ of all bounded real-valued, continuous functions on $X$ under various compact Hausdorff topologies. — Theorem 5.1: A necessary and sufficient condition that the abstract lattice $L$ be lattice-isomorphic to the lattice $C^*(X, T)$ for some compact Hausdorff space $(X, T)$ may be stated as follows: (1) The lattice $L$ contains a sublattice lattice-isomorphic to the set of all real numbers. (2) If $X=\{x, y, \cdots\}$ denote a suitable index set for the set $\phi=\{\varphi_x, \varphi_y, \cdots\}$ of all real lattice homomorpihsms on $L$, and $B(X)$ be the lattice of all real-valued functions defined and bounded on $X$, then, under the definition $f(x)=\varphi_x(f)$, the elements $f$ of $L$ determine distinct elements of $B(X)$. (3) Every sublattice of $B(X)$ containing properly all of the functions thus determined has a real lattice homomorphism other than the point real lattice homomorphism. This is a solution of Problem 81 of Birkhoff [Lattice theory, Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948, p. 176; MR **10**, 673]. *M. Novotný* (Brno).

**Goffman, Casper. Compatible seminorms in a vector lattice.** Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 536–538.

The author's first result is the observation that in every vector lattice with a total family of (order) bounded linear functionals there is a finest convex topology which is compatible with the order relation in the following sense: the topology is defined by semi norms $p$ such that $x\leqq y$ implies $p(x)\leqq p(y)$. One simply takes the topology defined by all such semi norms. Theorem 1 states this fact and some related observations. In theorem 2 the author shows that in a Banach lattice the order bounded and norm bounded linear functionals are identical and deduces that a vector lattice can be normed to be a Banach lattice in at most one way. {He is apparently unaware of the fact that the identity between order boundedness and norm boundedness is proved in Birkhoff's "Lattice theory" [Amer. Math. Soc. Colloq. Publ., v. 25, New York, 1940, p. 118, Th. 7.22; MR **1**, 325].}

*G. W. Mackey* (Cambridge, Mass.).

**Warner, Seth. Inductive limits of normed algebras.** Trans. Amer. Math. Soc. **82** (1956), 190–216.

Dans une algèbre (sur le corps réel ou le corps complexe) un ensemble $A$ est dit idempotent si $A^2\subset A$. Une algèbre localement multiplicativement convexe (en abrégé localement $m$-convex) est une algèbre munie d'une topologie localement convexe ayant un système fondamental de voisinages idempotents. Etant donnés une algèbre $E$, une famille $(E_\alpha)$ d'algèbres localement $m$-convexes, et pour tout $\alpha$ un homomorphisme $g_\alpha$ de $E_\alpha$ dans $E$, l'auteur définit sur $E$ la topologie (localement $m$-convexe) limite inductive de celles des $E_\alpha$ pour les homomorphismes $g_\alpha$ comme la plus fine des topologies localement $m$-convexes rendant continues les $g_\alpha$ (le plus souvent on a $E_\alpha\subset E$ et $g_\alpha$ est l'injection de $E$ dans $E$). Il peut se faire que la topologie ainsi définie sur $E$ soit strictement moins fine que la plus fine des topologies localement convexes rendant continues les $g_\alpha$, comme l'auteur le montre par un exemple. Toutefois il montre que les deux topologies coïncident lorsque

$E_\alpha CE$ est un idéal pour chaque $\alpha$, ou lorsque les $E_\alpha$ forment un ensemble totalement ordonné par inclusion, et que (dans les deux cas) $E$ est réunion des $g_\alpha(E_\alpha)$. L'auteur définit ensuite les notions relatives aux algèbres qui correspondent aux notions d'ensemble borné et d'espace bornologique: dans une algèbre localement $m$-convexe $E$, un ensemble $A$ est $i$-borné s'il existe $\lambda>0$ tel que $\lambda A$ soit contenu dans un ensemble borné idempotent; un homomorphisme de $E$ dans une algèbre localement $m$-convexe $F$ est $i$-borné si $f(A)$ est borné pour tout ensemble $i$-borné $A$ de $E$; enfin, $E$ est $i$-bornologique si tout homomorphisme $i$-borné est continu. Pour toute topologie localement $m$-convexe $\mathcal{J}$ sur $E$ il y a une topologie $i$-bornologique plus fine $\mathcal{J}^*$ dont les voisinages sont les ensembles convexes, équilibrés, absorbants, idempotents, et qui absorbent tous les ensembles $i$-bornés pour $\mathcal{J}$; dire que $\mathcal{J}^*=\mathcal{J}$ signifie que $\mathcal{J}$ est $i$-bornologique; les algèbres $i$-bornologiques sont limites inductives d'algèbres normées. Une limite inductive d'algèbres $i$-bornologiques est encore $i$-bornologique; la propriété analogue pour les produits infinis est équivalente à l'axiome d'Ulam affirmant la non-existence de mesures non nulles sur un ensemble infini, ne prenant que les valeurs 0 et 1, et nulles pour tout ensemble réduit à un point. Suivant Nachbin [Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 471–474; MR **16**, 156] l'auteur montre que l'algèbre $C(T)$ des fonctions numériques continues sur un espace complètement régulier $T$ est $i$-bornologique si et seulement si $T$ est un $Q$-espace de Hewitt (espace complet pour la moins fine des structures uniformes rendant uniformément continues toutes les fonctions de $C(T)$). Enfin l'auteur considère la classe des $P$-algèbres, savoir les algèbres localement $m$-convexes telles que l'ensemble des $x$ pour lesquels $\lim_{n\to\infty} x^n=0$ soit un voisinage de 0, et il met en rapport cette classe avec celles étudiées précédemment, lorsqu'il s'agit d'algèbres commutatives. Par exemple, si une algèbre commutative localement $m$-convexe $E$ est $i$-bornologique et si tout point est $i$-borné, $E$ est une $P$-algèbre; inversement, si $E$ (commutative) est une $P$-algèbre métrisable, elle est $i$-bornologique et tout point $y$ est $i$-borné; cette dernière propriété cesse d'être valable sans l'hypothèse de métrisabilité, comme l'auteur le montre par un exemple.

*J. Dieudonné* (Evanston, Ill.).

**See also:** Washnitzer, p. 69.

## *Banach Spaces, Banach Algebras*

**Willcox, Alfred B. Some structure theorems for a class of Banach algebras.** Pacific J. Math. **6** (1956), 177–192.

The structure space $S(R)$ of a Banach algebra $R$ is the set of maximal regular ideals with Stone-Jacobson topology. Let $R'$ be the Banach algebra obtained by adjoining an identity. If $S(R')$ is Hausdorff, then $R$ is a $GS$-algebra. Regular Banach algebras, $W^*$- and $AW^*$-algebras are $GS$-algebras. An algebra is strongly semi-simple if the intersection of the maximal regular ideals is 0. The author extends to strongly semi-simple $GS$-algebras many results proved for semi-simple regular Banach algebras by Šilov [Trudy Mat. Inst. Steklov. **21** (1947); MR **9**, 596 . The following are typical. For $x \varepsilon R$ and $M \varepsilon S(R)$, let $x(M)$ be the image of $x$ in the difference algebra $R/M$. For $F\subseteq S(R)$, let $\mathfrak{J}(F)$ be the set of $x$ such that $x(M)=0$ for $M$ in some open set containing $F$ and with compact complement. (1) If $F$ is closed and $I$ is an ideal with $F$ as the set of regular maximal ideals containing $I$, then $\mathfrak{J}(F)\subseteq I$. (2) If $I$ is an ideal, $x \varepsilon R$, and for each maximal regular ideal $M$ containing $I$ there is a compact set $C_M$ disjoint from $M$ and an $x_M \varepsilon I$ for which $x_M(M')=x(M')$ tor $M'$ in the complement of $C_M$, then $x \varepsilon I$. (3) If the closed ideal $I$ is contained only in a finite number of maximal regular ideals, then $I$ is an intersection of closed ideals each of which is contained in only one maximal regular ideal. The author defines $N$- and $N^*$-algebra, type $C$, and condition $(D)$ by natural modifications of Šilov's definitions [loc. cit.]. (4) If $R$ is also an $N^*$-algebra satisfying condition $(D)$ and $I$ is a closed ideal such that there is no perfect set contained in the boundary of the set of maximal regular ideals containing $I$, then $I$ is the intersection of the maximal regular ideals containing $I$. (5) If $R$ is also an $N^*$-algebra of type $C$ with identity, and for every $x \varepsilon R$ and bounded continuous real function $f$ on $S(R)$ there is a $y \varepsilon R$ for which $y(M)=f(M)x(M)$ for all $M \varepsilon S(R)$, then $R$ is an $N$-algebra. Subdirect sums of Banach algebras with only one maximal regular ideal are also discussed.

*J. A. Schatz* (Storrs, Conn.).

**Mori, Tuyosi; Amemiya, Ichirô; and Nakano, Hidegorô. On the reflexivity of semi-continuous norms.** Proc. Japan Acad. **31** (1955), 684–685.

Let $R$ be a semi-regular space. As shown by Nakano [J. Fac. Sci. Hokkaido Univ. Ser. I. **12** (1953), 87–104, Th. 8.4; MR **15**, 137] if a pseudo-norm on $R$ is continuous, then it is reflexive. The authors now show that the pseudo-norm must be reflexive if it is semi-continuous.

*I. Halperin* (Kingston, Ont.).

**★Люмис, Л. Введение в абстрактный гармонический анализ.** [L. Lyumis. (L. H. Loomis). Introduction to abstract harmonic analysis]. Izdat. Inostran. Lit., Moscow, 1956. 251 pp. 10.90 rubles.

Have English

A translation by D. A. Raĭkov of: An introduction to abstract harmonic analysis by L. H. Loomis [Van Nostrand, 1953; MR **14**, 883] with an introduction to the Russian edition by M. A. Naĭmark.

**Musielak, J.; and Orlicz, W. Linear functionals over the space of functions continuous in an open interval.** Studia Math. **15** (1956), 216–224.

If $C(a, b)$ is the set of bounded continuous functions in the open interval $(a, b)$ and if $a<t_1'<t_1''<b$, $t_n'\downarrow a$, $t_n''\uparrow b$, one can renorm the unit sphere of $C(a, b)$ by use of the formula $\|x\|^*=\sum_1^\infty 2^{-n}\|x\|_n$, where

$$\|x\|_n=\sup\{|x(t)|\ |t_n'\leq t\leq t_n''\}.$$

The renormed unit sphere is denoted by $K_s(a, b)$ (a Saks space) and a functional $\xi$ on $K_s(a, b)$ is called linear if $x_1$, $x_2$, $\lambda_1x_1+\lambda_2x_2 \varepsilon K_s(a, b)$ implies $\xi(\lambda_1x_1+\lambda_2x_2)=\lambda_1\xi(x_1)+\lambda_2\xi(x_2)$. The authors secure a representation for such functionals and a theorem concerning the convergence of sequences of such functionals. The basic tool is the Riesz representation of functionals on $C\langle t', t''\rangle$, the set of continuous functions on the closed interval $\langle t', t''\rangle$. The principal result reads as follows: A continuous linear functional $\xi$ on $K_s(a, b)$ may be written in the form $\xi(x)=\int_{a+}^{b-} x(t)dy(t)$ where (a) var $\{y(t)|a<t<b\}<\infty$, (b) $y$ is left continuous and (c) $y(t_0)=0$ for some $t_0 \varepsilon (t', t'')$.

*B. Gelbaum* (Minneapolis, Minn.).

**Ellis, H. W. On the basis problem for vector valued function spaces.** Canad. J. Math. **8** (1956), 417–422.

Continuing earlier work with Halperin [J. London Math. Soc. **31** (1956), 28–39; MR **17**, 646], the author derives theorems concerning bases in spaces $L^\lambda(S, X)$, $V^\lambda(S, X)$ of functions with values in a normed linear

space $X$ (not necessarily a Banach space). $\lambda$ represents a levelling length function (loc. cit.). As is well-known, the projections onto the finite partial sums in the general basis expansion are uniformly bounded in norm by some least constant $K$. The basis is then called $K$-regular. The principal result of the paper is: If $X$ has a $K'$-regular basis $\{x_i\}$ and if $\{\varphi_j(P)\}$ is a $K$-regular Haar or $\sigma$-Haar basis for $L^\lambda(S)$ (loc. cit.) then there is an ordering (constructively given) of the elements $x_i\varphi_j(P)$ of $L_\lambda(S, X)$ which makes the sequence a $3KK'$-regular basis in $L^\lambda(S, X)$. The proof is based in part on some lemmas which are straightforward extensions of some results of the reviewer [Thesis, Princeton, 1948, published in part in Ann. of Math. (2) **51** (1950), 26–36; MR **11**, 430].

*B. Gelbaum* (Minneapolis, Minn.).

**Hunt, G. A. Semi-groups of measures on Lie groups.** Trans. Amer. Math. Soc. **81** (1956), 264–293.

Let $G$ be a Lie group and let $G_e$ be its one-point compactification. Let $\{p_t\}$ $(0<t<\infty)$ be a family of probability measures on the Borel subsets of $G_e$ which forms a semi-group under convolution and satisfies the continuity condition $\lim_{t\downarrow 0} p_t(E)=1$ for any neighbourhood of the identity $e$ of $G$. Such a semi-group $\{p_t\}$ gives rise to a strongly continuous semi-group of linear operators $\{S_t\}$ on the Banach space $C=C(G_e)$ to $C$: $(S_tf)(g)=\int_{G_e} f(gh)p_t(dh)$. The Theorem 5.1 gives a characterisation of the infinitesimal generator $M$ of such semi-group $\{S_t\}$: $M$ is defined at least on $C^2(G_e)$ and has there the representation

$$(Mf)(g)=\sum a_iX_if(g)+\sum a_{ij}X_iX_jf(g) + \int_{G_e-\{e\}} [f(gh)-f(g)-\sum X_if(g)x_i(g)]\,\bar{G}(dh).$$

Here the symmetric positive semi-definite matrix $(a_{ij})$ is such that $\sum a_{ij}X_iX_j$ is independent of the choice of the basis $X_1, X_2, \cdots, X_d$ of the infinitesimal transformations of $G$; the positive measure $\bar{G}$ is such that $\int k(g)\bar{G}(dg)$ is finite for $C^2$ function $k(g)$ strictly positive on $G_e-\{e\}$ and behaves near $e$ like $\sum x_i^2$, $x_1(g)$, $x_2(g)$, $\cdots$, $x_d(g)$ denoting the local coordinates of $g \in G$ near $e$ such that $x_i(e)=0$ and $X_ix_j(e)=\delta_{ij}$. It is proved, in § 6, that the Lévy formula concerning the characteristic function of the infinitely divisible law may be obtained from the Theorem 5.1. In later §'s, the case where the continuity condition for $\{p_t\}$ is not satisfied is discussed in some detail. In such a case, $p_t$ converges, as $t$ decreases to 0, weakly to the Haar measure of a compact subgroup $K$ of $G$, and an analogue of Theorem 5.1 pertaining to the homogeneous space $G/K$ is sketched. *K. Yosida* (Tokyo).

**Kadison, Richard V. Operator algebras with a faithful weakly-closed representation.** Ann. of Math. (2) **64** (1956), 175–181.

The definition of a ring of operators (or $W^*$-algebra) calls for it to be weakly closed when acting on some Hilbert space. Already in the basic work of Murray and von Neumann it was apparent that this definition was technically awkward and gave rise to persistent investigations as to whether a given object was intrinsic or spatial. The problem of finding a useful characterization of $W^*$-algebras has since received repeated scrutiny. The reviewer will take this opportunity to summarize the various attacks. (1) Assumption of compactness of the unit sphere in a suitable topology. This was crucial in von Neumann's axioms [Mat. Sb. N.S. **1**(43) (1936), 415–484]. for Jordan algebras resembling $W^*$-algebras. An as yet unpublished theorem of Sakai simply postulates that the $C^*$-algebra, as a Banach space, be a dual; this of course makes the unit sphere compact in the weak* topology. The author's Theorem 3 is of this type. (2) Assumption of completeness relative to suitable functionals. This was the program of Steen [Proc. Cambridge Philos. Soc. **35** (1939), 562–578; MR **1**, 147], and Naimark [Uspehi Mat. Nauk (N.S.) **4** (1949), no. 4(32), 83–147; MR **11**, 186] has announced that he has plans for a similar investigation. (3) Assumption of the existence of projections and completeness of their lattice. This led Rickart [Ann. of Math. (2) **47** (1946), 528–550; MR **8**, 159] and the reviewer [ibid. **53** (1951), 235–249; MR **13**, 48] to a broader class of algebras ($AW^*$-algebras), but it is possible that the extra generality resides only in the center. (4) Postulation of order completeness on all self-adjoint elements (existence of a least upper bound for bounded directed sets). This is the novel attack in the present work, and a number of clever devices are needed to reach the goal. It seems likely that order completeness alone leads again to $AW^*$-algebras; in Theorem 1 the author assumes in addition the existence of a separating set of states "continuous" relative to order. In Theorem 2 he gives the simplification made possible by countability assumptions. *I. Kaplansky* (Princeton, N.J.).

**MacNerney, J. S. Continuous products in linear spaces.** J. Elisha Mitchell Sci. Soc. **71** (1955), 185–200.

This paper extends previous work of the author [Ann. of Math. (2) **61** (1955), 354–367; MR **16**, 716] on representation of functions by product integrals in an operator algebra. It also discusses continued fractions in the same general setting. *M. M. Day* (Seattle, Wash.).

**Rosenblum, Marvin. On the operator equation**

$$BX-XA=Q.$$

Duke Math. J. **23** (1956), 263–269.

In this interesting paper, the author considers an arbitrary Banach algebra $\mathcal{B}$ with identity $I$, considers fixed $A$ and $B \in \mathcal{B}$, and defines the bounded linear operator $T$ from $\mathcal{B}$ into $\mathcal{B}$ by $T(X)=BX-XA$. He uses the operational calculus of N. Dunford and A. Taylor to prove that $T$ possesses a bounded inverse $T^{-1}$ if $\sigma(B)$ and $\sigma(A)$ are disjoint, exhibiting $T^{-1}$ as a contour integral. This greatly strengthens some earlier theorems of Heinz [Math. Ann. **123** (1951), 415–438, esp. th. 5, p. 427; MR **13**, 471] and Rutherford [Akad. Wetensch. Amsterdam, Proc. **35** (1932), 54–59]. *F. H. Brownell.*

**See also:** Jones and Lumer, p. 7; Szeptycki, p. 22; Králik, p. 34; Massera, p. 42; Lax and Richtmyer, p. 48; Følner, p. 51; Goffman, p. 52; Warner, p. 52; Panafoti, p. 55; Rall, p. 72.

## *Hilbert Space*

**Dejon, Bruno. Über die stetigen Funktionen eines normalen Operators.** Ann. Univ. Sarav. **4** (1955), 200–205 (1956).

This is a short expository paper presenting directly the well-known spectral representation of the normed ring generated by a single bounded, normal operator on Hilbert space. *F. H. Brownell* (Seattle, Wash.).

**Panaĭoti, B. N. On the theory of linear singular equations in a unitary ring.** Akad. Nauk Azerbaĭdžan. SSR. Trudy Inst. Fiz. Mat. 3 (1948), 29–31. (Russian. Azerbaijani summary)

The author gives for linear equations in a unitary space results directly analogous to those of Z. I. Halilov [Dokl. Akad. Nauk SSSR (N.S.) **60** (1948), 1133–1136; MR **10**, 48] for a unitary ring. *F. V. Atkinson* (Canberra).

**Roberts, J. H. The rational points in Hilbert space.** Duke Math. J. 23 (1956), 489–491.

Homeomorphisms are constructed which (1) map the set of points in separable Hilbert space with all coordinates rational into a subset of the Euclidean plane and (2) map Hilbert space into the Hilbert cube. *D. C. Kleinecke.*

**See also:** Destouches, p. 95; Putnam, p. 98.

# TOPOLOGY

## *General Topology*

**Pták, Vlastimil. Compact subsets of convex topological linear spaces.** Czechoslovak Math. J. **4(79)** (1954), 51–74. (Russian. English summary)

In a complete uniform space, the closure of each precompact subset is compact; for metric spaces, this condition is equivalent to completeness. The author shows that the equivalence fails in more general spaces by constructing, in fact, a noncomplete locally convex topological linear space $X$ such that for each precompact $A\subset X$, the symmetric closed convex hull $A^{**}$ of $A$ is compact. This material is closely related to that of an earlier paper by the author [same J. **3(78)** (1953), 301–364; MR **16**, 262], and to the notion of $B$-completeness discussed therein. In extension of some results of that paper, the author shows here that if $E$ is a complete locally convex topological linear space and $E'$ its dual, then in $E'$ the topology of uniform convergence on compact sets is the weakest locally convex topology for $E'$ which agrees with the weak topology on every weakly compact subset of $E'$. *V. L. Klee* (Seattle, Wash.).

**Nagami, Keiô. Local properties of topological spaces.** Proc. Japan Acad. **32** (1956), 320–322.

A topological space $X$ has a property $P$ locally if every point in $X$ has a neighborhood with property $P$. The question of what local properties always hold globally was discussed by the reviewer in [Duke Math. J. **21** (1954), 163–171; MR **15**, 977]. The author states without proof some related results, and uses these (among other things) to generalize, from metric spaces to perfectly normal paracompact spaces, two theorems by Montgomery [Fund. Math. **25** (1935), 527–533, Th. 1, 2] on sets which are locally Borel or analytic. *E. Michael.*

**Mamuzić, Zlatko. Sur la topologie transitive d'une classe d'espaces ($\mathcal{V}$).** Bull. Soc. Math. Phys. Serbie **6** (1954), 63–73. (Serbo-Croatian summary)

Soit $E$ un espace, c'est-à-dire un ensemble quelconque d'éléments appelés points. L'A. introduit une topologie (généralisée) dans $E$ comme suit: il suppose donnée une fonction $f(a, b)$ appelée $M$-écart ou $M$-proximité définie sur $E\times E$ et prenant ses valeurs dans un ensemble donné quelconque $M$. Il suppose de plus qu'à chaque point $a\in E$, on a fait correspondre une famille $\mathcal{F}_a$ non vide de sous-ensembles de $M$ assujettie à la seule condition que $f(a, a)\in$ chaque ensemble de $\mathcal{F}_a$. Alors $E$ devient un espace ($\mathcal{V}$) de M. Fréchet en imposant le critère de contiguïté (C) suivant (que nous exprimons sous une forme légèrement différente de celle de l'A. mais équivalente): (C) Pour tout point $a\in E$ et tout ensemble $F\subset E$, on a: ($a$ contigu à $F$)$\rightleftharpoons$(pour tout $X\in\mathcal{F}_a$ il existe un $x\in F$ tel que $f(a, x)\in X$). — L'A. donne des formes équivalentes du critère (C); il établit des conditions pour que la topologie (généralisée) définie dans $E$ par (C) vérifie divers axiomes topologiques classiques; particulièrement il établit diverses conditions nécessaires et suffisantes pour que cette topologie soit transitive c'est-à-dire vérifie l'axiome $\alpha$ ($\bar{\bar{F}}=\bar{F}$). La conception de l'A. englobe comme cas particuliers les espaces à écart abstrait envisagés par divers auteurs (G. Kurepa, M. Fréchet, J. Colmez, A. Appert). — L'A. généralise les espaces $\mathcal{E}[M]$ de Kurepa [C. R. Acad. Sci. Paris **203** (1936), 1049–1052] en introduisant les espaces $E(M_{\mathcal{V}})$ définis comme étant les espaces $E$ topologisés par (C) comme plus haut et où de plus $M$ est un espace ($\mathcal{V}$) et où, pour chaque point $a\in E$, la famille $\mathcal{F}_a$ est, dans l'espace $M$, la famille des voisinages de $f(a, a)$. L'A. démontre que dans les espaces $E(M_{\mathcal{V}})$ le critère (C) peut être remplacé par le critère équivalent $O^3$ suivant dû à G. Kurepa (loc.cit.): $O^3$. Pour tout point $a\in E$ et tout ensemble $F\subset E$, on a: ($a$ contigu à $F$)$\rightleftharpoons$($f(a, a)$ contigu à $f(a, F)$), où on pose: $f(a, F)$=ensemble des $f(a, x)$ pour $x\in F$. *A. Appert* (Angers).

**Ellis, David. A theorem on description adequacy.** Publ. Math. Debrecen **4** (1956), 180–183.

A class of nets $N$ in a space $S$ is said to adequately describe the topology of $S$ provided the following proposition holds: If $T\subset S$ and $p\in S$, then $p$ is an accumulation point of $T$ if and only if there is a net in $T-\{p\}$ which is a member of $N$ and which has limit $p$. It is well-known that any one of the following classes of nets in a topological space adequately [d]escribes the topology of the space: (a) The set of (cou[nt]able) sequences in a locally separable space; (b) the set of phalanxes in a topological space; (c) the set of topophalanxes in a topological space; (d) the set of ultraphalanxes in a topological space; (e) the set of all nets in a topological space; (f) the set of ultranets in a topological space. [For (a), see R. Vaidyanathaswamy, Treatise on set topology, part I, Indian Math. Soc., Madras, 1947; MR **9**, 367. For (b), (c), (d), see J. W. Tukey, Convergence and uniformity in topology, Princeton, 1940; MR **2**, 67. For (e) and (f), see J. L. Kelley, Duke Math. J. **17** (1950), 277–283; MR **12**, 194.]

The author shows in this paper that every net has a subnet which is a phalanx. From this, and known results, it follows that (b), (c), (d), (e), and (f), are equivalent. It is also shown that it is a simple matter "to construct an adequate class of nets in a topological space directly from the mapping whose existence is a form of the axiom of choice; namely, the selection operator." *D. W. Hall.*

**Iséki, Kiyoshi; and Miyanaga, Yasue. A theorem on paracompact spaces.** Proc. Japan Acad. **32** (1956), 396–398.

The following theorem is proved: Let $f$ be a real-valued function defined on a paracompact space $X$. If, for every point $x\in X$, the oscillation $\omega(x; f)$ of $f$ at $x$ is $\leqq\alpha$, then for any $\varepsilon>0$, there exists a real-valued continuous function $g$ on $X$ such that $|f(x)-g(x)|\leqq\alpha+\varepsilon$ for all $x\in X$. The proof

uses a partition of unity subordinate to a locally finite open covering of $X$. *Ky Fan* (Notre Dame, Ind.).

**Iséki, Kiyoshi. On the Lebesgue property in uniform spaces.** Publ. Math. Debrecen 4 (1956), 239–241.

Results previously published by the author in Proc. Japan Acad. 31 (1955), 220–221, 270–271, 441–442, 524–525, 618–619 [MR 17, 389]. *E. A. Michael.*

**Hayashi, Yoshiaki. A note on dimension functions of separable metric spaces.** Math. Japon. 3 (1955), 161–162.

A dimension function, $d$, is defined to be a function which assigns to each separable metric space a non-negative integer and which also has the properties: a) $d(E^n)=n$ for $E^n$ $n$-dimensional Euclidean space; b) if $R$ is homeomorphic to $R'$, then $d(R)=d(R')$; c) if $R\supset R'$, then $d(R)\geqq d(R')$. Consider the following propositions: (1) If $R=R_1\cup R_2$, then $d(R)\leqq d(R_1)+d(R_2)$; (2) if $R=R_1\times R_2$, then $d(R)=d(R_1)+d(R_2)$; (3) if $R$ is the countable union of closed sets $R_i$ with $d(R_i)\leqq n$ for each $i$, then $d(R)\leqq n$. The author shows that no dimension function satisfies both (1) and (2), exhibits a dimension function which he claims satisfies (2) and (3), and points out that Menger's rational dimension is a dimension function satisfying (1) and (3). *Haskell Cohen.*

**Berstein, I. Remarques sur un théorème de F. J. Dyson relatif à la sphère.** Fund. Math. 43 (1956), 89–94.

As a generalization of a theorem of Dyson [Ann. of Math. (2) 54 (1951), 534–536; MR 13, 450] the author proves the following theorem: Let $E$ be a connected, locally connected, unicoherent, compact metric space, let $T: E\to E$ be a topological involution without fixed point and let $f$ be a continuous real-valued function on $E$. Then for any number $d$, $0<d\leqq \inf \varrho(x, T(x))$ there exist $a, b\in E$ such that $\varrho(a, b)=d$ and $f(a)=f(b)=f(T(a))=f(T(b))$, where $\varrho$ is the distance function on $E$. {Note: From a result of the reviewer [ibid. 62 (1955), 271–283, p. 281; MR 17, 289] it is easily seen that this theorem holds when $(E; T)$ is a $T$-space of $B$-index $\geqq 2$ in the sense of the reviewer and $\varrho$ is a continuous real valued function on $E\times E$ such that $\inf \varrho(x, T(x))>0$. Notice that if $E$ is connected, unicoherent and compact Hausdorff, then $(E; T)$ is a $T$-space of $B$-index $\geqq 2$.} *C. T. Yang.*

**Nagami, Keiô. On some types of polyhedra.** Proc. Japan Acad. 32 (1956), 245–247.

This paper deals with simplicial $CW$-complexes which are $n$-full (i.e. every set of at most $n$ vertices determines a simplex). The principal result is that if $K$ is such a complex, and $X$ a metric space with a closed subset $A$ such that $\dim (X-A)\leqq n$, then every continuous $f: A\to K$ can be extended continuously over $X$. *E. A. Michael.*

**Weier, Josef. Lokale Wesentlichkeit von Abbildungen und eindimensionaler Zusammenhang von Polyedern.** Arch. Math. 7 (1956), 201–203.

Let $P$ be a finite connected euclidean polyhedron, $V$ a set which is open in $P$, $b$ a point of $V$ and let $g$ be a mapping $P\to P$ which, in $\bar{V}$, admits $b$ as its only fixed point. The point $b$ is called inessential if $g$ can be continuously modified on $\bar{V}$ so as to be rid of any fixed point in $\bar{V}$. An index is attached to $b$ and it is shown that if $b$ is inessential, the index is zero. If the index is zero and $P$ is 1-connected at $b$ (i.e. if $W-b$ is connected for some neighborhood $W$ of $b$), then $b$ is inessential. It is also shown how an isolated fixed point of index $\zeta$ can be "decomposed" into a finite number of fixed points whose index-sum is $\zeta$. *P. A. Smith* (New York, N.Y.).

**Molnár, J. Über eine Verallgemeinerung auf die Kugelfläche eines topologischen Satzes von Helly.** Acta Math. Acad. Sci. Hungar. 7 (1956), 107–108. (Russian summary)

This note is devoted to showing that if any family of closed subsets of the 2-sphere is such that (i) the product of each two members of the family is connected, (ii) the product of each three is non-empty, and (iii) the 2-sphere is not covered by any four of the sets, then the family has a non-empty product. Induction, anchored by a family of exactly four members (the assertion is then a consequence of the Helly theorem) establishes the theorem for finite families, and the theorem of F. Riesz is applied to complete the proof. It is observed that a theorem due to C. V. Robinson [Amer. J. Math. 64 (1942), 260–272, Th. 4; MR 3, 300] follows immediately from this result. *L. M. Blumenthal* (Columbia, Mo.).

**Dowker, C. H.; and Hurewicz, W. Dimension of metric spaces.** Fund. Math. 43 (1956), 83–88.

The sequential dimension for a metric space $X$, $(dsX)$ is the least integer $n$ such that there exists a sequence $\{\alpha_i\}$ of locally finite coverings, each of order $n$, with mesh $\alpha_i\to 0$ as $i\to\infty$, and such that the closure of each member of $\alpha_{i+1}$ is contained in some member of $\alpha_i$. It is shown that $dsX=\dim X$ where $\dim X$ is the usual covering dimension of $X$. A new proof is incidentally given that $\dim X=\operatorname{Ind} X$ (the inductive dimension defined in terms of separation of closed sets). [See also M. Katětov, Dokl. Akad. Nauk SSSR (N.S.) 79 (1951), 189–191; MR 15, 145; and K. Morita, Math. Ann. 128 (1954), 350–362; MR 16, 501.] *Haskell Cohen* (Baton Rouge, La.).

**Ford, L. R. Jr.; and Fulkerson, D. R. Maximal flow through a network.** Canad. J. Math. 8 (1956), 399–404.

The determination of the maximal steady flow in certain transportation networks can be formalized as follows. A graph $G$ is a finite collection of vertices $a, b, \cdots$, and arcs joining them. A positive capacity is assigned each arc of $G$. A chain joining the vertices $a$ and $h$ is a set of distinct arcs which can be arranged as $(ab)$, $(bc)$, $\cdots$, $(gh)$, where the vertices are distinct. A chain flow from $a$ to $b$ is a chain from $a$ to $b$ and a nonnegative number $h$ representing the flow along the chain from $a$ to $b$. A flow from $a$ to $b$ is a collection of chain flows from $a$ to $b$ such that the sum of the $h$'s for the chains that contain each arc does not exceed the capacity of that arc. A collection $D$ of arcs which meets every chain from $a$ to $b$ is a disconnecting set with value $v(D)$ equal to the sum of the capacities of its members; an irreducible disconnecting set is a cut. A graph is $ab$-planar if, together with arc $(ab)$, it is planar; i.e. can be drawn in the plane without crossing of arcs. Theorem 1: the maximum flow from $a$ to $b$ is the minimum of $v(D)$ over all disconnecting sets $D$. Theorem 2: an $ab$-planar graph contains a chain joining $a$ and $b$ that meets every cut precisely once. A specialized computing procedure for obtaining maximal flows (a problem in the general category of linear programming) can be based on these theorems. *T. E. Harris.*

**Harary, Frank. On the number of dissimilar line-subgraphs of a given graph.** Pacific J. Math. 6 (1956), 57–64.

The author establishes two enumeration formulae for

graphs in which no two nodes are joined by more than one edge. They are obtained as special cases of a general combinatorial theorem due to G. Pólya [Acta Math. **68** (1937), 145–254].

The first formula gives the number of dissimilar line-subgraphs, i.e. subgraphs including all the vertices, having a given number of edges, for any graph $G$. The second deals with graphs having the same vertices as $G$. Two such graphs are "similar $G$-rooted graphs" if there is an isomorphic mapping of one into the other which is an automorphism of $G$. The formula gives the number of dissimilar $G$-rooted graphs with a given number of edges.

*W. T. Tutte* (Toronto, Ont.).

**See also:** Guérindon, p. 8; Reisel, p. 9; Gillman and Henriksen, p. 9; Hochschild, p. 10; Jakubík, p. 22; Goffman, p. 52; Warer, p. 52; Roberts, p. 55; Washnitzer, p. 69.

## *Algebraic Topology*

**Araki, Shôrô. On Steenrod's reduced powers in singular homology theories.** Mem. Fac. Sci. Kyûsyû Univ. Ser. A. 9 (1956), 159–173.

Following Steenrod's method [Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 213–217, 217–223; MR **14**, 1005, 1006] for defining the reduced $p$th power operations, the crucial step is in the definition of an equivariant chain mapping

$$\Phi: W(\Pi)\otimes C\to C\otimes\cdots\otimes C \quad (p\text{-fold tensor product}),$$

where $\Pi$ is the cyclic group of order $p$, $W(\Pi)$ is an acyclic, $\Pi$-free complex, and $C$ is a chain complex. Steenrod defined $\Phi$ in case $C=C(X)$ is the cellular chain complex of a cell complex $X$. In this paper $\Phi$ is defined for the simplicical singular complex, normalized simplicial singular complex, simplicial singular complex normalized at the top, normalized cubical singular complex, cubical singular complex normalized at the top, of any topological space $X$. It is shown that within any of these categories any two equivariant maps $\Phi$ define the same cohomology operations. For each of these categories an inductive definition of $\Phi$ is given starting from Steenrod's formula for $\Phi$ on the one-dimensional part of $C$. In the case $p=2$ (the squaring operations) an explicit formula is given for $\Phi$ in the normalized cubical singular theory. Finally it is shown that if the cohomology groups obtained from these various singular theories are identified in a certain natural way, the reduced power operations coincide. *N. Stein.*

**Thomas, Emery. A generalization of the Pontrjagin square cohomology operation.** Proc. Nat. Acad. Sci. U.S.A. 42 (1956), 266–269.

L'auteur définit, pour chaque $p$ premier, une „puissance de Pontrjagin":

$\mathfrak{P}_p: H^n(K; Z/pmZ)\to H^{pn}(K; Z/p^2mZ)$ ($m$ entier, $n>0$), qui, pour $p=2$, coïncide avec le „carré de Pontrjagin" défini par J. H. C. Whitehead; pour $p$ impair, $\mathfrak{P}_p$ est nul si $n$ est impair, et

$$\mathfrak{P}_p(uv)=\mathfrak{P}_p(u)\mathfrak{P}_p(v) \text{ (cup produit)}, \quad \eta_*\mathfrak{P}_p(u)=u^p,$$

où $\eta_*$ est l'application naturelle $H^{pn}(K; Z/p^2mZ)\to H^{pn}(K; Z/pmZ)$. De plus, en cohomologie relative, $\mathfrak{P}_p$ s'annule sur l'image du cobord $\delta$. Ces opérations ne sont pas triviales (cf. l'espace projectif complexe de grande dimension paire).

Soit $\Pi$ un groupe abélien de type fini; l'auteur généralise son produit de Pontrjagin et définit, pour chaque entier $t>0$ et chaque $n$ pair, une opération $\mathfrak{P}_t: H^n(K;\Pi)\to H^{tn}(K;\Gamma_t(\Pi))$, où $\Gamma(\Pi)$ désigne l'algèbre graduée définie par Eilenberg et MacLane [Ann. of Math. (2) **60** (1954), 49–139; MR **16**, 391]. On a, pour $u$, $v\in H^n(K;\Pi)$,

$$\mathfrak{P}_r(u)\mathfrak{P}_s(u)=(r,s)\mathfrak{P}_{r+s}(u) \quad ((r,s)\text{: coefficient binomial}),$$

$$\mathfrak{P}_t(u+v)=\sum_{r+s=t}\mathfrak{P}_r(u)\mathfrak{P}_s(v),\ \mathfrak{P}_0(u)=1,\ \mathfrak{P}_1(u)=u.$$

*H. Cartan* (Paris).

**Aleksandrov, P. S. Nondualizability of Betti groups based upon finite coverings.** Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 5–6. (Russian)

**Aleksandrov, P. S. Correction to the paper "The nondualizability of Betti groups based upon finite coverings".** Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 357. (Russian)

In connection with the general duality theorems proved by the author and by Sitnikov a simple example is presented to show that the Čech groups $H_f$ based on finite open coverings ate not "dualized", i.e. there exist two sets in the $n$-sphere with homeomorphic complements, but with non-isomorphic $H_f$-groups. For $n=2$ the two sets are: $A$, a half-open segment, and $A'$, a disc from the boundary of which one point has been removed. The group $H_f^2(A')$ is shown to be non-zero via mappings into the 2-sphere (Dowker's theorem): Represent $A'$ as the upper half plane $y\geqq 0$ minus the point at infinity; map all lines $y=$integer into a single point $p$ of the 2-sphere, and map the interior of each strip $n<y<n+1$ homeomorphically onto the complement of $p$; this map is essential in the sense of uniform homotopy. In the first note $A'$ was, by mistake, defined as a different, unsuitable set.

*H. Samelson* (Ann Arbor, Mich.).

**Murasugi, Kunio. Covering spaces and the invariant $k$.** Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A. **5** (1955), 49–51.

Let $X$ be a simply-connected topological space whose homotopy groups $\pi_i(X)$ vanish in dimensions $n<i<q$ and $i>q$. Let $(\tilde X, p, X)$ be the Cartan-Serre-Whitehead fibre-space such that $\tilde X$ is $(n-1)$-connected and $p_*:\pi_i(\tilde X)\cong\pi_i(X)$, $i\geqq n$. Then the space $\tilde X$ defines an Eilenberg-Maclane invariant $\tilde k_{q+1}=\tilde k\in H^{q+1}(\pi_n, n; \pi_q)$. The author relates this to the Postnikov invariant $k_{q+1}=k$ of $X$; indeed, his assertion is that, if $p_\#: H^{q+1}(X,\pi_q)\to H^{q+1}(\tilde X,\pi_q)$ is induced by $p$, then $p_\# k'=\tilde k'$, where $k'$, $\tilde k'$ are geometrical realizations of $k$, $\tilde k$.

However, it appears to the reviewer that the statement requires some modification (and this is borne out by the author's proof). We may attach cells of dimensions $q+1$, $q+2$, $\cdots$, to $X$ and form a space $B$ such that $\pi_i(B)=0$, $i\geqq q$; we may then find a fibre-space $(\tilde B, \tilde p, B)$ such that $\tilde B$ is $(n-1)$-connected $\tilde p_*:\pi_i(\tilde B)\cong\pi_i(B)$, $i\geqq n$, $\tilde B\supseteq\tilde X$ as a closed subset and $\tilde p|\tilde X=p$. Then there are realizations of $k$, $\tilde k$ in $H^{q+1}(B,\pi_q)$, $H^{q+1}(\tilde B,\pi_q)$ and we may consider the conjectural equality

$$(1) \qquad \tilde p_\# k'=\tilde k';$$

from this would follow $p_\# i_\# k'=i_\# k'$, where $i$: $X\to B$, $\tilde i$: $\tilde X\to\tilde B$ are inclusions.

The author's proof of (1) is based on the result, published by Nakaoka [Proc. Japan Acad. **30** (1954), 363–368; MR **16**, 506], relating the Eilenberg-Maclane and

Postnikov invariants to transgression in a certain fibre-space. Namely, in this case, if we replace the inclusion $i$ by an equivalent fibration then the fibre $F$ is a $K(\pi_q, q)$ and if $b \in H_q(F, \pi_q)$ is the basic cohomology class, then $\tau b = -k'$, where $\tau$ is the transgression. With a similar notation, $\tilde{\tau}\tilde{b} = -\tilde{k}'$. The author produces a map $p_1 : \tilde{F} \to F$ such that $\tilde{\tau} p_{1\#} = \tilde{p}_{\#}\tau$, so that it remains only to prove that $p_{1\#} b = \tilde{b}$. {This is claimed to follow from the fact that $p_1$ is a homotopy equivalence, but the reviewer considers that a conclusive argument requires a closer examination of the map $p_1$.} *P. J. Hilton* (Manchester).

**Kuratowski, K. Sur une méthode de métrisation complète de certains espaces d'ensembles compacts.** Fund. Math. 43 (1956), 114–138.

In an earlier note [Bull. Acad. Polon. Sci., Cl. III. 3 (1955), 75–80; MR 16, 1139] the author obtained a condition for a set $S$, on which a class of convergent sequences is defined, to be metrizable with a complete metric such that metric convergence of sequences and the original convergence coincide. This condition was then applied to $r$-regular convergence of closed subsets of a compact metric space, thereby obtaining a theorem of Begle [Duke Math. J. 11 (1944), 441–450; MR 6, 95] in the homology case and a generalization of a theorem of Borsuk [Fund. Math. 41 (1955), 168–202; MR 16, 946] in the homotopy case. The present paper gives a more elaborate and systematic treatment of the same subject, including proofs previously omitted, with previous applications generalized from closed subsets of compact spaces to compact subsets of complete spaces, and with two new applications. For example, the latest version of Borsuk's [ibid.] theorem asserts that the space of non-empty, compact, $LC^n$ subsets of a complete metric space can be metrized with a complete metric to yield homotopy-$n$-regular convergence. *E. Michael.*

**Kudo, Tatsuji. A transgression theorem.** Mem. Fac. Sci. Kyûsyû Univ. Ser. A. 9 (1956), 79–81.

Let $(E, p, B)$ be a fibre space in the sense of Serre, and let $(E_r)$ be its cohomology spectral sequence over $Z_p$. Let $\alpha \in E_{r+1}^{a,b}$ and define an integer $\theta(\alpha) = \theta$ so that

$$d_{\rho+1}\varkappa_{\rho+1}^{r+1}(\alpha) = 0 \ (r \leq \rho < \theta);$$
$$d_{\theta+1}\varkappa_{\theta+1}^{r+1} \neq 0.$$

$\alpha$ is said to be transgressive if $\theta(\alpha) \geq b = DF(\alpha)$. If $\alpha$ is transgressive, there exists a $\beta \in E_2^{a+b+1,0}$ such that

$$(*) \qquad d_{b+1}\varkappa_{b+1}^{r+1}(\alpha) = \varkappa_{b+1}^2(\beta).$$

The following theorem is proved. Let $\alpha \in E_2^{0,2k} \approx H^{2k}(F, Z_p)$ be transgressive and let $\beta$ be defined by (*). Let $\Delta_p$ be the Bockstein operator associated with the exact sequence

$$0 \to Z_p \to Z_{p^2} \to Z_p \to 0.$$

Then $\mathcal{P}_p^k(\alpha) = \alpha^p$ and $\beta \cdot \alpha^{p-1}$ are transgressive, and

$$d_{2pk+1}\varkappa_{2pk+1}^2(\alpha^p) = \varkappa_{2pk+1}^2(\mathcal{P}_p^k\beta),$$
$$d_{2(p-1)k+1}\varkappa_{2(p-1)k+1}^2(\beta \cdot \alpha^{p-1}) = \varkappa_{2(p-1)k+1}^2(-\Delta_p\mathcal{P}_p^k\beta).$$

*N. Stein* (Ithaca, N.Y.).

**Kudo, Tatsuji; Mukohda, Shunji; and Saito, Shiroshi. Reduction formulas for Steenrod's $D_i$ in the cubic singular cohomology theory.** Mem. Fac. Sci. Kyûsyû Univ. Ser. A. 9 (1956), 101–110.

The Steenrod reduced $p$th powers were defined for a cell complex $K$ [Ann. of Math. (2) 56 (1952), 47–67; MR 13, 966] by means of homomorphisms

$$D_i : C_q(K) \to C_{q+i}(K \times K \times \cdots \times K) \ (p\text{-fold product}).$$

Here $C_j(X)$ denotes the group of $j$-dimensional cellular chains of the cell complex $X$. In this paper it is shown how to introduce the reduced powers in a general space by defining the $D_i$ for cubical singular chains. They are defined inductively starting from the formula Steenrod has given [Proc. Nat. Acad. Sci. U.S.A. 39 (1953), 217–223; MR 14, 1006] for $D_i$ on a one-dimensional chain.

Denoting the dual cochain homomorphism again by $D_i$ the following theorem concerning filtration in a fibre space is proved. If $f_j \in A^{*a_j}$ $(j = 1, 2, \cdots, n)$, we have

$$D_i(f_1 \times f_2 \times \cdots \times f_n) \in A^{*\max(a_1+a_2+\cdots+a_n-i,0)}.$$

*N. Stein* (Ithaca, N.Y.).

**Weier, Joseph. Contribution to the theory of transformations of $(n+1)$-spheres into $n$-spheres.** Rev. Mat. Hisp.-Amer. (4) 16 (1956), 72–81. (Spanish)

Let $S$, $T$ be spheres in a Euclidean space such that $\dim S = (\dim T) + 1 \geq 4$, $P_0$ a point of $T$, $g : S \to T$ a mapping defined by $g(p) = P_0$ for every point $p \in S$, and $f : S \to T$ any continuous mapping. Let $F$, $G$ denote the classes of those continuous mappings from $S$ into $T$ which are homotopic to $f$ and $g$ respectively. The author raises the question whether there exist mappings $f^1 \in F$ and $g^1 \in G$ such that $f^1(p) \neq g^1(p)$ for every point $p \in S$, and proves that the answer is in the negative if $f$ is an essential mapping from $S$ onto $T$. Next, the author considers a pair of mappings $f^1 \in F$ and $g^1 \in G$ and denotes by $\zeta$ the minimum for all such pairs $f^1$, $g^1$, of the number of those points $p \in S$ where $f^1(p) = g^1(p)$, and proves that $\zeta$ is finite. The proofs are based on standard devices in homotopy theory. *T. Radó* (Columbus, Ohio).

**Nakagawa, Ryôsuke. On cohomotopy loops.** Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A. 5 (1955), 53–61.

The homotopy classification problem of mappings of a complex $X$ into an $n$-sphere $S^n$ was investigated by the reviewer [Ann. of Math. (2) 50 (1949), 158–173; MR 10, 393] for the case where $\dim X < 2n-2$ and by C. Chen [ibid. 51 (1950), 238–240; MR 11, 380] for the case where $\dim X = 2n-2$. In this paper, the author proves that, for some $n$, an analogous result holds without any dimensional restriction on $X$. These values of $n$ are determined by the condition that there exists a mapping $\alpha$ of $S^n \times S^n$ into $S^n$ of type (1, 1), or equivalently, there exists a mapping $\beta$ of $S^{2n+1}$ into $S^{n+1}$ of Hopf invariant 1. *S. T. Hu.*

**James, I. M. The suspension triad of a sphere.** Ann. of Math. (2) 63 (1956), 407–429.

Using the results of two earlier papers [Ann. of Math. (2) 62 (1955), 170–197; 63 (1956), 191–247; MR 17, 396, 1117] the author continues his study of the suspension triad. In particular he obtains results which apply to the suspension triad which arises in studying the suspension homomorphism from a sphere to the next higher dimensional spheres.

Let $S^n$ denote the $n$-dimension sphere, and let $(S^{n+1}; E_+, E_-)$ be the triad such that $E_+$ is the northern hemisphere and $E_-$ the southern hemisphere of $S^{n+1}$. For this triad $S^{n+1} = E_+ E_-$, and $S^n = E_+ E_-$. Further let

$$h : \pi_r(S^{n+1}; E_+, E_-) \to \pi_r(S^{2n+1})$$

denote the natural homomorphism defined by the author

in the paper cited above. The author proves three main theorems. Theorem. Let $n$ be odd. Then

$$h:\pi_r(S^{n+1}; E_+, E_-)\simeq\pi_r(S^{2n+1}).$$

Theorem. Let $n$ be even. Then the order of the kernel of $h$ is odd, and has no prime factor greater than $r/m$. Also the index of the subgroup

$$h\pi_r(S^{n+1}; E_+, E_-)\subset\pi_r(S^{2n+1})$$

is odd, and has no prime factor greater than $r/2n$. Theorem. Let $n$ be even, and let $p$ be an odd prime number. If $r\leqq 2pn-2$, then the $p$-primary component of the kernel of

$$h:\pi_r(S^{n+1}; E_+, E_-)\to\pi_r(S^{2n+1})$$

is a direct summand of $\pi_r(S^{n+1}; E_+, E_-)$. If $r\leqq 2pn-4$, then this direct summand is isomorphic to the direct sum

$$\pi_r(S^{pn})\otimes Z_p+\operatorname{Tor}(\pi_r(S^{pn+1}), Z_p).$$

Using these theorems the author obtains many explicit results on the homotopy groups of spheres. The first two theorems are particularly useful in studying the 2-primary component of the homotopy of spheres, which at present seems to be the most difficult part of the study.

*J. C. Moore* (Princeton, N.J.).

**Sugawara, Masahiro. *H*-spaces and spaces of loops.** Math. J. Okayama Univ. **5** (1955), 5–11.

The author generalizes a construction of the reviewer [Comment. Math. Helv. **28** (1954), 278–287; MR **16**, 389] in which topological groups were shown to be *H*-isomorphic with the loop spaces of their classifying spaces; the groups are replaced by *H*-spaces, and the principal bundles by bundles which are contractible in a containing space. We describe its application to a special case. Let $p$: $S_{15}\to S_8$ be the familiar Hopf fiber map, based upon Cayley numbers; attach a 16-cell $e_{16}$ to $S_8$ by means of $p$, to obtain a space $Y$; then there is a map of $S_7$ (identified with the unit sphere of the Cayley numbers) into the loop space of $Y$ which is an *H*-homomorphism and induces an isomorphism of the homotopy groups in dimensions $\leqq 20$. From this it is concluded, by a method analogous to that of the reviewer for quaternions, making use of the known structure of certain homotopy groups of spheres, that the Cayley numbers are not homotopy-commutative, i.e., that the map $(x, y)\to xyx^{-1}y^{-1}$ of $S_7\times S_7\to S_7$ is not null-homotopic. *H. Samelson* (Ann Arbor, Mich.).

**Federer, Herbert. A study of function spaces by spectral sequences.** Trans. Amer. Math. Soc. **82** (1956), 340–361.

The author uses the technique of exact couples to study the homotopy groups of function spaces. He first generalizes the notion of exact couple to include the case in which certain groups in the couple are non-abelian. Now assume $K$ is a *CW*-complex on the topological space $X$, and $Y$ is an arcwise connected topological space which is simple in all dimensions and has an abelian fundamental group. Let $v: X\to Y$ be a continuous map and let $v_q=v|K_q$, where $K_q$ denotes the $q$-dimensional skeleton of $K$. Let $Y^{K_q}$ denote the function space of maps of $K_q$ into $Y$ with the compact-open topology and let $U_q$ be the arc component of $v_q$ in $Y^{K_q}$. The restriction on map $r_q$: $U_q\to U_{q-1}$ is a fibre map in the sense of Serre with fibre $F_q=r_q^{-1}\{v_{q-1}\}$. Let $A_{p,q}=\pi_p(U_q, v_q)$, $C_{p,q}=\pi_p(F_q, v_q)$ for $p\geqq 0$ and $A_{p,q}=C_{p,q}=\{0\}$ for $p<0$. Let $A=\oplus_{p,q} A_{p,q}$ and $C=\oplus_{p,q} C_{p,q}$ and define an exact couple $\nabla(K, Y, v)$

$$\begin{tikzcd} A \arrow[rr, "f"] & & A \arrow[dl, "h"] \\ & C \arrow[ul, "g"] & \end{tikzcd}$$

as follows: $f|A_{p,q}: A_{p,q}\to A_{p,q-1}$ is induced by $r_q$ for $p\geqq 0$ and is trivial if $p<0$; $g|A_{p,q}: A_{p,q}\to C_{p-1,q+1}$ is the boundary operator of the fibre map $r_{q+1}$ for $p\geqq 1$ and is trivial for $p<1$; $h|C_{p,q}: C_{p,q}\to A_{p,q}$ is induced by the inclusion $F_q\subset U_q$ for $p\geqq 0$ and is trivial for $p<0$.

A map $\gamma$: $C_{p,q}\to C^q(K, \pi_{p+q}(Y))$ is defined and is shown to be univalent in general, onto if $p\geqq 1$, and a homomorphism if $p\geqq 1$. It is then shown that for $p\geqq 0$ and $q\geqq 0$, $\gamma$ induces a monomorphism $\gamma^*$: $C_{p,q}^{(1)}\to H^q(K, \pi_{p+q}(Y))$ which is onto for $p\geqq 1$, where $C_{p,q}^{(1)}$ denotes the term in the first derived couple $\nabla^{(1)}(K, Y, v)$.

If $p\geqq 1$ the restriction operation defines maps $\varrho_q$: $(Y^X, v)\to(Y^{K_q}, v_q)$ which carry the arc component of $v$ in $Y^X$ into $U_q$. Let $G_{p,q}$ be the kernel of the induced homomorphism $\varrho_{q*}$: $\pi_p(Y^X v)\to A_{p,q}$. Then there is a normal chain

$$\pi_p(Y^X, v)=G_{p,-1}\supset G_{p,0}\supset G_{p,1}\supset\cdots.$$

If $\dim X=Q<\infty$, then $G_{p,Q}=\{0\}$ and

$$G_{p,q-1}/G_{p,q}\approx G_{p,q}^{(n)} \text{ for } n\geqq \max(q, Q-q).$$

Suppose $Y$ is an Eilenberg-Maclane space $K(\Pi, m)$, $X$ is a finite dimensional cellular space, and $v\in Y^X$. As an application of his techniques the author obtains the following result which has also been obtained by A. Heller and R. Thom:

$$\pi_p(Y^X, v)\approx H^{m-p}(X, \Pi) \text{ for } p\geqq 1.$$

As another example let $Y=S^m$ be an $m$-sphere, $m\geqq 1$ and let $R$ be the field of rational numbers. Let $\mu'\in H^m(S^m, R)$ be the cohomology class of a cocycle with value 1 on the unique $m$-cell of $S^m$ and let $v\in(S^m)^X$. The author obtains an exact sequence for $p\geqq 2$

$$H^{m-p-1}(X, R)\xrightarrow{\cup v^*(\mu')}H^{2m-p-1}(X, R)\to\pi_p((S^m)^X, v)\otimes R$$
$$\to H^{m-p}(X, R)\xrightarrow{\cup v^*(\mu')}H^{2m-p}(X, R).$$

*N. Stein* (Ithaca, N.Y.).

**Adams, J. F. On the cobar construction.** Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 409–412.

For any space $X$ which is arcwise connected and simply connected, let $\Omega(X)$ denote the space of loops in $X$. In a previous paper [Comment. Math. Helv. **30** (1956), 305–330; MR **17**, 1119], the present author and P. Hilton gave a construction for obtaining the homology ring $H_*(\Omega X)$ from the cellular structure of $X$ in case $X$ is a *CW*-complex with a single vertex and no 1-cells. In the present paper, the author gives a construction for obtaining $H_*(\Omega X)$ from the knowledge of the group of singular chains, $C(X)$, and the homomorphism $C(X)\to C(X)\otimes C(X)$ induced by the diagonal map, $X\to X\times X$. This construction is called the "co-bar construction" because in some sense it is dual to the bar construction of Eilenberg and MacLane. It associates with every chain complex $C$ which has no 1-cells and a single vertex for which an

"associative" diagonal map, $C \to C \otimes C$, is given, a chain ring, $F(C)$, having a boundary operator. Algebraically, $F(C)$ is the free associative algebra generated by the cells of $C$ of positive dimension. The boundary operator on $F(C)$ is defined by means of the boundary operator on $C$ and the diagonal map. The main theorem asserts that if $C=C(X)$ as above, then $H_*(\Omega X)$ is naturally isomorphic to $H_*(F(C))$. *W. S. Massey* (Providence, R.I.).

**Curtis, M. L. The covering homotopy theorem.** Proc. Amer. Math. Soc. 7 (1956), 682–684.

The author proves that if $E$, $B$ are metric, and $(E, B, p)$ is a locally trivial fiber space, then $(E, B, p)$ has the covering homotopy property with respect to all topological spaces. {Reviewer's remark: This result is also contained in Hurewicz, Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 956–961; MR **17**, 519.} *J. Dugundji.*

**Milnor, John. On the immersion of $n$-manifolds in $(n+1)$-space.** Comment. Math. Helv. **30** (1956), 275–284.

Let $M$ be a closed oriented differentiable $n$-manifold of class $C^1$. An immersion of $M$ in an Euclidean $(n+1)$-space $E^{n+1}$ is a map $f: M \to E^{n+1}$ of class $C^1$ whose Jacobian matrix has rank $n$ at all points. If $f$ is one-one, it is called an imbedding. After $E^{n+1}$ is oriented, the orientations of $M$ and $E^{n+1}$ define at each point $x \in M$ a unit normal vector $N(x)$. If $N(x)$ is considered as a point of the unit sphere $S^n$, this defines the normal map $N: M \to S^n$. This paper studies the question of the possible degrees of the normal map.

For $n$ even a classical result of H. Hopf says that the degree is equal to one-half of the Euler-Poincaré characteristic of $M$. Hopf also proved that if, for $n$ odd, $M$ can be immersed in $E^{n+1}$ with degree $d$, it can also be immersed with any degree $d'$ which is congruent to $d$ modulo 2. It follows that, for $n$ odd, the sphere $S^n$ can be immersed in $E^{n+1}$ with arbitrary odd degree. These results are given new proofs in this paper. A sphere bundle argument shows that if $M$ can be immersed in $E^{n+1}$ with normal degree zero, then $M$ is parallelizable. Hence if, for odd $n$, $M$ is not parallelizable and can be immersed in $E^{n+1}$ at all, then it can be immersed with arbitrary odd degree, but cannot be immersed with even degree. Also, if $n$ is a dimension for which $S^n$ is parallelizable, then every closed orientable $n$-manifold immersed in $E^{n+1}$ is parallelizable. On the other hand, if $n$ is an odd dimension for which $S^n$ is not parallelizable and if $M$ is sphere-like and parallelizable and can be immersed in $E^{n+1}$, then it can be immersed with arbitrary even degree, but cannot be immersed with odd degree. In the case of immersion of spheres it is proved by an explicit construction that $S^3$ can be immersed in $E^4$ with even degree. The first unsettled case is whether $S^7$ can be immersed in $E^8$ with even degree.

For an imbedding of $M$ in $E^{n+1}$ it is proved that the degree $d$ of the normal map satisfies: $d \equiv \frac{1}{2}\Sigma(M)$, mod 2, $|d| \leq \frac{1}{2}\Sigma(M)$, where $\Sigma(M)$ is the sum of Betti numbers of $M$. *S. Chern* (Chicago, Ill.).

**Dold, Albrecht. Erzeugende der Thomschen Algebra $\mathfrak{N}$.** Math. Z. 65 (1956), 25–35.

Let $\Omega$ and $\mathfrak{N}$ denote the cobordism algebras of R. Thom [Comment. Math. Helv. **28** (1954), 17–86; MR **15**, 890]. The elements of $\Omega$ are equivalence classes of oriented, compact, differentiable manifolds, and those of $\mathfrak{N}$ are equivalence classes of non-oriented manifolds. Each of these rings is graded. Thom proved [loc. cit.] that $N$ is a polynomial algebra in a countable infinitude of indeterminates $x_i$ over the field of integers modulo 2. The variable $x_i$ can be represented by a properly chosen manifold of dimension $i$. The index $i$ takes on all positive integral values such that $i+1$ is not a power of 2. Thom actually gave representative manifolds for $x_i$ if $i$ is even or if $i=5$.

In the present paper the author gives representative manifolds $P(i)$ for $x_i$ for the remaining values of $i$, i.e., $i$ odd and $>5$, $i \neq 2^k-1$. The manifold $P(i)$ has a two-fold covering which is the product of a sphere and a complex projective space.

It turns out that the manifold $P(i)$ is orientable. If it is given an orientation, it represents an element of the ring $\Omega$; this element is of order 2. This gives some additional information about the ring $\Omega$. *W. S. Massey.*

**Jaworowski, J. W. Theorem on antipodes for multivalued mappings and a fixed point theorem.** Bull. Acad. Polon. Sci. Cl. III. 4 (1956), 187–192.

This paper extends some theorems of Borsuk [Fund. Math. **20** (1933), 177–190] on single valued maps to multiple valued maps, the approach being by means of the Vietoris mapping theorem [for related results using this approach see a paper by Eilenberg and the reviewer, Amer. J. Math. **68** (1946), 214–222; **8**, 51]. One of the corollaries giving a good idea of the content is stated as follows: Let $F$ be an upper semi-continuous mapping which assigns to every $x$ in $S_n$ a compact acyclic subset $F(x)$ of $E_n$. Then there exist antipodal points $x_0$ and $y_0$ in $S_n$ such that $F(x_0)$ intersects $F(y_0)$. *D. Montgomery.*

**Weier, Josef. Bemerkungen zu einem Homotopieproblem.** Arch. Math. 7 (1956), 100–106.

Let $P$ be a euclidean polyhedron in euclidean space $R^n$. Let $F=(f^\tau)$, $0 \leq \tau \leq 1$ be a homotopy of $P$ into itself such that for each $\tau$, $f^\tau$ has only a finite number of fixed points. Then $F$ is said to be isolated. Let $\alpha_0 < \alpha_1$ be numbers in the closed interval $[0, 1]$. The open interval $(\alpha_0, \alpha_1)$ is said to be a simple interval of $F$ if 1) for all numbers $\sigma$, $\tau$ such that $\alpha_0 < \sigma < \tau < \alpha_1$, the maps $f^\sigma$ and $f^\tau$ have the same number of essential fixed points. and 2) if $\beta_0 < \beta_1$ are in $[0, 1]$ and $(\alpha_0, \alpha_1)$ is a proper subinterval of $(\beta_0, \beta_1)$ then 1) does not hold when $\alpha_i$ is replaced by $\beta_i$. Let $q \in P$ and $\alpha \in [0, 1]$. Suppose that for any open set $U \subset P$ containing $q$ and any $\varepsilon > 0$ there exists a $\beta \in [0, 1]$ such that $|\beta - \alpha| < \varepsilon$ and $f^\beta$ has at least two essential fixed points in $U$. Then $(q, \alpha) \in R$ is called a branch point of $F$. The set $S \subset R^{n+1}$ of all points $(p, \tau)$ such that $p$ is a fixed point of $f^\tau$ is called the singular image of $F$. $F$ is finite if $S$ is empty or a finite homogeneous one-dimensional polyhedron.

Theorem. A finite homotopy $F$ has only a finite number of simple intervals and a finite number of branch points. Given $f$ and $f'$ as above, a number $\zeta$ is defined by the following conditions: a) There exists an isolated homotopy of $f$ to $f'$ which has exactly $\zeta+1$ simple intervals; b) each isolated homotopy of $f$ to $f'$ has at least $\zeta+1$ simple intervals. A number $\eta$ is defined by the following conditions: a′) There exists an isolated homotopy $\eta$ of $f$ to $f'$ which has $\eta$ branch points; b′) each isolated homotopy of $f$ to $f'$ has at least $\eta$ branch points. It is shown that $\zeta$ and $\eta$ are topologically invariant.

Theorem. If $\eta=0$, then there exists an isolated homotopy $(g^\tau)$ of $f$ to $f'$ such that for $0 \leq \tau \leq 1$ each fixed point of $g^\tau$ is inessential. Theorem. If $f$ and $f'$ are sufficiently close then $\zeta=\eta=0$. Theorem. If $\zeta=0$, then $\eta=0$. Theorem. $\zeta \leq \eta$. *N. Stein* (Ithaca, N.Y.).

**Conner, P. E. Concerning the action of a finite group.** Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 349–351.

Denote by $G$ a finite group of transformations on the compact space $X$. Let $\pi: X \to X/G$ denote the natural map of $X$ onto the space of orbits. The following results are announced. (1) The homomorphism $\pi^*$ maps the cohomology group $H^*(X/G; C)$, $C$ the complex numbers, isomorphically onto the subgroup of $H^*(X; C)$ consisting of all $x$ with $g^*(x)=x$ for all $g \in G$. If $X$ is a differentiable manifold (with or without boundary) on which $G$ acts differentiably, or if $X$ is a finite cell complex on which $G$ acts cellwise, or if $X$ is cohomology locally connected for the prime $p$ and $r=$order $G=p^j$, then $(G, X)$ is called admissible. Then (2) if $(G, X)$ is admissible then $\chi(X/G)=\sum_G \chi(F_g)/r$, where $\chi$ is the Euler characteristic and $F_g$ is the set of elements of $X$ fixed under $g$. Finally (3) if a compact connected abelian Lie group $L$ operates continuously on a closed manifold $M$, then $\chi(F)=\chi(M)$, $F$ the set of points of $X$ fixed under all $g \in G$. *E. E. Floyd.*

**de Lyra, C. B. Minimal complexes and maps.** Bol. Soc. Mat. São Paulo **7** (1952), 85–98 (1954).

The author proves the existence of a minimal subcomplex of the cubical singular complex of a topological space. This proof parallels the proof of the existence of a minimal subcomplex of the total singular complex of a space by Eilenberg and Zilber [Ann. of Math. (2) **51** (1950), 499–513; MR **11**, 734]. After this the author uses the techniques developed to prove that if $X$ and $Y$ are arcwise connected spaces, $n$ a positive integer, $f: X \to Y$ a map which induces isomorphisms between the homotopy groups of $X$ and those of $Y$ in dimensions less than or equal to $n$, and the homotopy groups of $Y$ are zero in dimensions greater than $n$, then the minimal complex of $Y$ is determined by that of $X$ and the integer $n$. Actually knowledge of this theorem is just one step in the study of the minimal complex of $X$ by means of its Postnikov system [Dokl. Akad. Nauk SSSR (N.S.) **66** (1949), 161–164; **67** (1949), 427–430; **71** (1950), 1027–1028; MR **11**, 451, 676], and the theorem can be proved easily from this point of view. *J. C. Moore* (Princeton, N.J.).

See also: Berstein, p. 56; Washnitzer, p. 69.

# GEOMETRY

## *Geometries, Euclidean and other*

**Shepperd, J. A. H. Transitivities of betweenness and separation and the definitions of betweenness and separation groups.** J. London Math. Soc. **31** (1956), 240–248.

In einer $bz\leq$ totalgeordneten Menge wird eine 3-stellige Zwischenrelation $B(a, b, c)$ definiert durch $a\leq b\leq c$ oder $c\leq b\leq a$. Eine 4-stellige Trennungsrelation $S(a, b, c, d)$ wird definiert durch $a\leq b\leq c\leq d$ oder $d\leq c\leq b\leq a$ oder eine der hieraus durch zyklische Vertauschung entstehenden Aussagen. Verf. gibt eine axiomatische Charakterisierung dieser Zwischen- und Trennungsrelationen, wobei die Unabhängigkeit durch Gegenbeispiele gezeigt wird. *P. Lorenzen* (Bonn).

**Sydler, J.-P. Sur les tétraèdres équivalents à un cube.** Elem. Math. **11** (1956), 78–81.

If a tetrahedron can be dissected into a finite number of pieces which can be reassembled to form a cube, the dihedral angles of the tetrahedron must all be commensurable with $\pi$; therefore a regular tetrahedron is not "equivalent" to a cube [M. Dehn, Math. Ann. **55** (1902), 465–478]. Three one-parameter families of tetrahedra equivalent to a cube may be constructed as follows [M. J. M. Hill, Proc. London Math. Soc. **27** (1896), 39–53]. The first kind is one of the six congruent tetrahedra into which the general rhombohedron is dissected by its three planes of symmetry. In other words, it is a tetrahedron $OABC$ whose edges $OA$, $AB$, $BC$ represent three vectors of equal magnitude making three equal angles with one another. This tetrahedron is symmetrical by a half-turn about the line $MN$, where $M$ and $N$ are the mid-points of the edges $OC$ and $AB$. The second and third kinds of Hill tetrahedra are $OCNA$ and $ABMO$. The author has found four new tetrahedra which are equivalent to a cube although they are not Hill tetrahedra. The first of the four is the trirectangular tetrahedron $X_4$ of the reviewer's "Regular polytopes" [Methuen, London, 1948, p. 192; MR **10**, 261]. The second is the sum of two congruent $X_4$'s; the third and fourth are the sum and difference of two similar $X_4$'s. *H. S. M. Coxeter.*

**Legrain-Pissard, [N.]. Sur les homographies de l'espace.** Bull. Soc. Roy. Sci. Liège **25** (1956), 209–221.

L'auteur étudie dans l'espace projectif à trois dimensions la décomposition des homographies en produits d'homographies biaxiales harmoniques et d'homologies harmoniques. Successivement chaque type d'homographie spatiale est considéré, et pour chacun on précise la décomposition en question en indiquant encore de combien de façons elle peut être réalisée. Les définitions utilisées dans cette note sont celles de M. M. Godeaux et Rozet [Leçons de Géométrie projective, 2ième éd., Sciences et Lettres, Liège, 1952; MR **13**, 861]. *M. Decuyper.*

**Naumann, Herbert. Eine affine Rechtwinkelgeometrie.** Math. Ann. **131** (1956), 17–27.

In an affine plane the following are taken as the trivial axioms on orthogonality of lines: O1. If $g\perp h$, then $h\perp g$. O2. If $g\perp h$ and $h\|k$, then $g\perp k$. O3. Through every point of a line $g$ there is exactly one line $h$ with $g\perp h$. Together with these axioms the author studies the consequences of Incidence Theorem A: If $P_i$ ($i=0, 1, 2, 3$) and $P_i'$ ($i=0, 1, 2, 3$) are the vertices of non-degenerate quadrangles and five of the relations $P_iP_k\perp P_i'P_k'$ hold, then the sixth also holds. From this the affine theorem of Desargues follows. In terms of the coordinatizing division ring $K$, the orthogonality may be described in terms of an automorphism $a\to\bar{a}$ of $K$ and a particular fixed element $\varphi$ such that $\varphi\bar{\bar{a}}=a\varphi$ for every $a\in K$. Lines $y=ax+s$ and $y=bx+t$ are orthogonal if and only if $a\varphi\bar{b}=-1$. The most general collineations preserving orthogonality are found. Ordered and Pascalian systems are studied and their relation to the theorem on the orthocenter of a triangle. *Marshall Hall, Jr.* (Columbus, Ohio).

**Barlotti, Adriano. Un'osservazione sulle $k$-calotte degli spazi lineari finiti di dimensione tre.** Boll. Un. Mat. Ital. (3) **11** (1956), 248–252.

Sei $S_{3,q}$ ein linearer, endlicher, 3-dimensionaler Raum

der Ordnung $q$. Eine $k$-Kalotte des $S_{3,q}$ sei eine Menge von $k$ Punkten, dieses Raumes, von denen keine 3 auf einer Geraden liegen. Die Maximalzahl von $k$, für die in $S_{3,q}$ eine $k$-Kalotte existiert, ist $k=q^2+1$. Zu der Frage, für welche Werte von $k$ mit $k \leqq q^2$ jede $k$-Kalotte in einer $(q^2+1)$-Kalotte des $S_{3,q}$ enthalten ist, gibt Verf. einen Beitrag durch den Beweis der Sätze: Für $q \geqq 7$ ist jede $(q^2-h)$-Kalotte des $S_{3,q}$ mit $0 \leqq h \leqq q-7$ in einer $(q^2+1)$-Kalotte enthalten. Für $q=3$ und $q=5$ ist jede $q^2$-Kalotte in einer $(q^2+1)$-Kalotte enthalten. Vergl. hierzu die analogen Untersuchungen von B. Segre in endlichen Ebenen [Ann. Mat. Pura Appl. (4) **39** (1955), 357–379; MR **17**, 776].

*R. Moufang* (Frankfurt am Main).

**Schütte, Kurt. Ein Schliessungssatz für Inzidenz und Orthogonalität.** Math. Ann. **129** (1955), 424–430.

The author finds a geometric necessary and sufficient condition that an affine-metric plane possess a fundamental metric form, so that coordinates may be set up in terms of an involutorial skew-field and perpendicularity may be characterized by vanishing of a scalar product. The condition is a closure theorem (Schliessungssatz) in the sense of Reidemeister [Grundlagen der Geometrie, Springer, Berlin, 1930]: In two complete quadrangles with vertices $P_0, P_1, P_2, P_3$ and $P_0', P_1', P_2', P_3'$ and sides $g_i=P_0P_i$, $h_1=P_2P_3$, $h_2=P_3P_1$, $h_3=P_1P_2$ and similarly $g_i'$, $h_i'$ ($i=1, 2, 3$), if any five of the six relations $g_i \perp h_i'$, $g_i' \perp h_i$ ($i=1, 2, 3$) hold, then so does the sixth.

The same statement with perpendicularity replaced by parallelism is equivalent to the affine form of the Pappus-Pascal theorem. Both are consequences of the theorem concerning the concurrence of altitudes of a triangle, and both imply the Desargues theorem.

Finally, the closure theorem is a necessary condition that the plane may be embedded in an affine-metric space. It therefore plays the same role in affine-metric geometry that the Desargues theorem plays in Projective geometry. [See R. Baer, Linear algebra and projective geometry, Academic Press, New York, 1952; MR **14**, 675; and H. Lenz, Math. Ann. **128** (1954), 363–372; MR **16**, 7.]

*S. Gorn* (Philadelphia, Pa.).

**Schütte, Kurt. Die Winkelmetrik in der affin-orthogonalen Ebene.** Math. Ann. **130** (1955), 183–195.

In a Desarguesian affine plane with a symmetric orthogonality relation between lines and no isotropic lines, let the angular measure between $a$ and $b$, $W(a, b)$, be defined as the ratio $SA_{ba}:SA$ where $S$ is the intersection of $a$ and $b$, $A$ on $a$, $A \neq S$, $AA_b \perp b$, and $A_bA_{ba} \perp a$. In order that this definition be independent of the choice of $A$ or of the order of $a$ and $b$ it is necessary that the following closure theorem, a special case of the Pappus-Pascal Theorem, hold: (W) If $A$, $A_{ba}$, $B_a$ are points on $a$, and $A_b$, $B$, $B_{ab}$ are points on $b$ such that $A_bA_{ba} \| BB_a \perp a$ and $AA_b \| B_aB_{ab} \perp b$, then $AB \| A_{ba}B_{ab}$.

Let the closure theorem which is equivalent to the characterization of perpendicularity by the vanishing of a scalar product be designated by (M) (see preceding review).

When (M) holds, a necessary and sufficient condition for (W) is that the square of any vector is in the center of the coordinate skew-field. If, in addition, the Fano axiom holds (The diagonals of a rectangle intersect), the condition is that the coordinate domain be either a commutative field or a quaternion-division algebra. In this last case commuting the factors of a scalar product produces the conjugate quaternion.

There is a one-one correspondence between the lines $x$ through $S$ having equal (but not right) angles with the fixed line $a$ through $S$ and the elements of the multiplicative group (in the skew-field) of all $\varepsilon$ for which $\varepsilon^{\mathfrak{J}}\varepsilon=1$, where $\mathfrak{J}$ is the involutorial anti-automorphism of the field. Thus there are only two such $x$ if and only if $\mathfrak{J}$ is the identity, in which case the theorem of concurrence of the altitudes of a triangle holds.

To 'rotate' an angle $xa$ about the intersection $S$ to $b$, designated by $xa \to bx'$, one uses the following construction: For any $P \neq S$ on $x$, take $P_a$ on $a$ and $P_b$ on $b$ such that $PP_a \perp a$ and $PP_b \perp b$; then $x'$ is determined through $S$ by $x' \perp P_aP_b$. This construction is independent of the choice of $P$ by Desargues' theorem, and obviously also yields $xb \to ax'$. (M) implies: if $xa \to bx'$, then $x'a \to bx$; this means that the mapping $x \to x'$ determined by $a$ and $b$ is involutorial. (W) implies: if $xa \to bx'$, $ya \to by'$, and $x' \perp y$, then $x \perp y'$. This theorem is equivalent to the closure theorem W' which is also a special case of the Pappus-Pascal theorem: (W'): If $P_a$, $S$, $Q_a$ and $P$, $P_b$, $Q$ are distinct points, respectively, on the lines $a$ and $c$ such that $PP_a \| QQ_a \perp a$, $P_aP_b \| SQ$, and $P_bS \perp c$, then $PS \| P_bQ_a$. (W) is equivalent to W' by use of Desargues' theorem. If $xa \to bx'$ and $ya \to by'$, then $W(x, y)=W(x', y')$; thus $xa \to bx'$ implies $W(a, x)=W(b, x')$.

For sensed angles we then find that, unless $a \perp b$, $\sphericalangle(a, b)=\sphericalangle(b, a)$ only if the involution $\mathfrak{J}$ is not the identity. To define sensed angles in the usually accepted meaning therefore requires that the theorem of the concurrence of altitudes holds. This theorem follows from the assumption that angular equality, as defined by the perpendicular construction, is transitive.

The author also, discusses line reflections and the generated group of displacements and symmetries, and the relationship between the closure theorems and, for example, the theorem of the three concurrent line reflections.

*S. Gorn* (Philadelphia, Pa.).

**Villa, Mario. Progressi recenti nella teoria delle trasformazioni puntuali.** Confer. Sem. Mat. Univ. Bari no. **10** (1955), 19 pp. (1956).

Summary of some results on point-transformations between two projective planes.

*E. Bompiani* (Rome).

**Gergely, Eugen. Sur les cônes et coniques de la géométrie de Lobatchevsky-Bolyai.** Acad. R. P. Romîne. Bul. Ști. Secț. Ști. Mat. Fiz. **7** (1955), 1025–1034. (Romanian. Russian and French summaries)

**Böhm, Johannes. Über Spezialfälle bei der Inhaltsmessung in Räumen konstanter Krümmung.** Wiss. Z. Friedrich-Schiller-Univ. Jena **5** (1955/56), 157–164.

The author investigates the trigonometry of spherical and hyperbolic simplexes along the lines of Schläfli [Ges. Math. Abh., Bd. 1, Birkhäuser, Basel, 1950, pp. 227–302; MR **11**, 611], Schoute [Mehrdimensionale Geometrie, Bd. 1, Göschen, Leipzig, 1902, pp. 267–286] and the reviewer [Bull. Calcutta Math. Soc. **28** (1936), 123–144]. In particular, he elucidates Schläfli's generalization of Napier's chain of five related right-angled triangles, and extends the reviewer's work on the Lobačevskiĭ function [Quart. J. Math. Oxford Ser. **6** (1935), 13–29]. The paper is illustrated with remarkable diagrams.

*H. S. M. Coxeter* (Toronto, Ont.).

**Fejes Tóth, L. On the volume of a polyhedron in non-Euclidean spaces.** Publ. Math. Debrecen 4 (1956), 256–261.

The author investigates the general quadrirectangular tetrahedron (or orthoscheme) in spherical or hyperbolic space [cf. Coxeter, Quart. J. Math. Oxford Ser. 6 (1935), 13–29, p. 21]. He obtains a comparatively simple expression for its volume as a definite integral, and illustrates this by applying it to the characteristic tetrahedron of the star-polytope {3, 3, 5/2} [Coxeter, Regular polytopes, Methuen, London, 1948, p. 284; MR 10, 261]. Knowing that 14400 such tetrahedra cover the whole spherical space $d$ times, where $d$ is the density of this star-polytope, he uses Simson's formula to obtain the approximation $d\sim 190.6$, in remarkable agreement with the exact value $d=191$. *H. S. M. Coxeter.*

See also: Goodstein, p. 2; Richardson, p. 102; Molnár, p. 56; Blank, p. 64.

## *Convex Domains, Integral Geometry*

**Gran Olsson, R. Über Porenvolumen und Porenziffer in der Erdbaumechanik und die verschieden dichte Packung von Kugeln. II.** Norske Vid. Selsk. Forh., Trondheim 29 (1956), 22–23.

This gives an extension of the author's first table [same Forh. 28 (1955), 96–99; MR 17, 521] and deals with the same three kinds of packing in different soils. *M. E. Wise* (Penarth).

**Weinberger, H. F. An isoperimetric inequality for the $N$-dimensional free membrane problem.** J. Rational Mech. Anal. 5 (1956), 633–636.

Für die Eigenwertaufgabe $\Delta u+\mu u=0$ im $N$-dimensionalen Raum mit der Randbedingung $\partial u/\partial n=0$ ist bekanntlich der niedrigste Eigenwert gleich Null. Für den ersten von Null verschiedenen Eigenwert wird im Anschluß an das zugehörige Rayleigh-Ritz'sche Minimalproblem bewiesen, daß ein kugelförmiger Bereich das Maximum liefert. Für den Beweis wird der Brouwer'sche Fixpunktsatz herangezogen. *P. Funk* (Wien).

**Fejes Tóth, L. Über die dünnste Horozyklenüberdeckung.** Acta Math. Acad. Sci. Hungar. 7 (1956), 95–98. (Russian summary)

It is known that the density of any covering of the hyperbolic plane by equal circles exceeds that of the covering by horocycles circumscribed about the faces of the tessellation $\{\infty, 3\}$ [Fejes Tóth, same Acta 4 (1953), 111–114; MR 15, 341; Coxeter, ibid. 5 (1954), 263–274, p. 265; MR 17, 523]. The author now proves that the density of any covering of the hyperbolic plane by horocycles is likewise $\geq 2\sqrt{(3)}/\pi$. *H. S. M. Coxeter.*

**Mack, C. On clumps formed when convex laminae or bodies are placed at random in two or three dimensions.** Proc. Cambridge Philos. Soc. 52 (1956), 246–250.

This note continues the author's work on counting laminae when they fall at random onto a plane in large enough concentrations for some or many of them to overlap. He first acknowledges a correction, pointed out by S. A. Roach, to his earlier result [same Proc. 50 (1954), 581–585; MR 16, 267] which did not allow for every possible kind of clump that could be formed by three laminae or more. The new formulae correspond to clumps of plane projections of convex bodies in space that cannot penetrate one another. *M. E. Wise* (Penarth).

**Fejes Tóth, L. Characterisation of the nine regular polyhedra by extremum properties.** Acta Math. Acad. Sci. Hungar. 7 (1956), 31–48. (Russian summary)

The author considers a polyhedron having $v$ vertices, $e$ edges, $f$ faces, circum-radius $R$, in-radius $r$, and surface-area $F$. In an earlier work [Canad. J. Math. 2 (1950), 22–31; MR 11, 386] he established the inequalities

$$\frac{R}{r}\geq \tan\frac{\pi}{p}\tan\frac{\pi}{q},\ \frac{F}{r^2}\geq e\sin\frac{2\pi}{p}\left(\tan^2\frac{\pi}{p}\tan^2\frac{\pi}{q}-1\right),$$

where $p=2e/f$, $q=2e/v$, with equality only for the regular polyhedra $\{p, q\}$. He now proves that these inequalities remain valid for star-polyhedra, provided $f$ and $v$ are weighted with the densities of the faces and vertex figures, respectively (e.g., $f$ is doubled when the faces are pentagrams).

Turning his attention to higher spaces, he proves that, among those convex polytopes which are topologically isomorphic with a regular polytope, the regular one has the smallest value for $R/r$. Moreover, among the star-polytopes of density 191 having tetrahedral cells, five at each edge, the regular {3, 3, 5/2} has the smallest value for $R/r$. *H. S. M. Coxeter* (Toronto, Ont.).

**Masotti Biggiogero, Giuseppina. La Geometria Integrale.** Rend. Sem. Mat. Fis. Milano 25 (1953–54), 164–231 (1955).

A very complete exposition of the fundamental concepts and results on which the integral geometry in the plane and in the space is based, with a short account of the recents developments of the theory. The paper is mainly expository, though it contains some new integral formulas of the author referring to convex domains on the plane and on the space. The paper ends with an almost exhaustive bibliography on the subject. *L. A. Santaló.*

**Stoka, Marius. La mesure d'un ensemble de variétés dans un espace. $R_n$.** Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 7 (1955), 903–937. (Romanian. Russian and French summaries)

The author gives explicit expressions for the invariant measure of a continuous group of transformations and does applications to some particular cases. The measure of the group of similitudes on the plane allows to calculate the measure of sets of circles; the measure of the projective group gives a measure for sets of conics; the measure of rotations plus homotheties about a point in $E_3$ gives the measure of sets of circles on the sphere. {The author does not use the method of differential forms of E. Cartan, which probably should simplify the calculations [see, e.g., the reviewer's book Introduction to integral geometry Hermann, Paris, 1953; MR 15, 736].} *L. A. Santaló.*

See also: Oshio, p. 75.

## *Differential Geometry*

**Lane, N. D.; and Sherk, F. A. Differentiable points of arcs in conformal 3-space.** Canad. J. Math. 8 (1956), 105–118.

This paper extends to the conformal 3-space the classification of the differentiable points in the conformal plane

given previously [Canad. J. Math. 5 (1953), 512–518; MR 15, 251]. Besides the pencils $\gamma_1$ and $\gamma_2$ of the tangent and osculating circles at a point $p$ of an arc (continuous image of a closed real interval) $A$, the pencils $\sigma_1$, $\sigma_2$, and $\sigma_3$ of the tangent, osculating and surosculating spheres are defined in a similar manner and relations between them investigated. $p$ is called a differentiable point of $A$ if there exists a unique surosculating sphere $S(\sigma_3)$ of $A$ in $p$, which implies the existence of the osculating circle $C(g_2)$. At an interior differentiable point all the spheres of $\sigma_0-\sigma_1(\sigma_1-\sigma_2, \sigma_2-\sigma_3)$ support $A$ at $p$ or they all intersect. The classification given takes into account the various corresponding possibilities and the eventual degeneration of $C(\gamma_2)$ and $S(\sigma_3)$ into points. Examples of each of the types are provided by the curves $x=t^m$, $y=t^n$, $z=t^r$ and $x=t^m$, $y=t^n$ $z=t^r \sin t^{-1}$, if $0<|t|\leq 1$, $=0$ if $t=0$ where $m$, $n$, $r$ are positive integers and $m<n<r$. *C. Y. Pauc.*

**Lane, N. D.; and Scherk, P. Characteristic and order of differentiable points in the conformal plane.** Trans. Amer. Math. Soc. 81 (1956), 358–378.

The paper under review presupposes the knowledge of a preceding paper by the same authors [Canad. J. Math. 5 (1953), 512–518; MR 15, 251]. Theorem 1: Let $p$ be a differentiable interior point of an arc $A$. Suppose that $p$ has the characteristic $(a_0, a_1, a_2)$ or $(a_0, a_1, a_2)_0$. Then the (cyclic) order of $p$ is not less than $a_0+a_1+a_2$. Hint to the proof: The osculating circle $C(p)$ is first approximated by a tangent circle, then the latter by a non-tangent circle through $p$, and finally that circle by one which does not contain $p$. Definition: An arc $A$ is strongly differentiable at $p$ if the following conditions are satisfied: Condition I'. Let $R\neq p$, $Q\to R$. If two distinct points $u$ and $v$ converge on $A$ to $p$, then the circle $C(u, v, Q)$ always converges. Condition II'. $C(t, u, v)$ converges if the three mutually distinct points $t$, $u$, $v$ converge on $A$ to $p$. Theorem 2: Every point of an open arc of order three satisfies condition I'. Theorem 3: Let $p$ be an end point (defined by continuity) of an open arc $A_3$ of order three. Then $A_3\cup p$ is strongly differentiable at $p$. Definition: A point $p$ of an arc $A$ is conformally elementary if a neighbourhood of $p$ exists on $A$ which is decomposed by $p$ into two one-side neighbourhoods or order three. Theorem 4: Let $p$ be a differentiable conformally elementary point of an arc $A$. Then the order of $p$ is $=a_0+a_1+a_2$. Theorem 5: Let $p$ be a conformally elementary of an arc $A$. Then $A$ is strongly differentiable at $p$ if and only if it is differentiable at $p$ and $a_0=a_1=1$. {Remarks. The reviewer's notion of "point on a curve" and its carrier [Les méthodes directes en géométrie différentielle, Hermann, Paris, 1941; MR 7, 67] can provide a more rigorous formulation for the paper above. Condition II', if the limit circle of $C(t, u, v)$ is not reduced to a point, implies condition I'. See pamphlet just quoted § 2}. *C. Y. Pauc* (Lafayette, Ind.).

**Blank, Ya. P. On the problem of N. G. Čebotarev concerning generalized translation surfaces.** Har'kov Gos. Univ. Uč. Zap. 40=Zap. Mat. Otd. Fiz.-Mat. Fak. i Har'kov Mat. Obšč. (4) 23 (1952), 103–112 (1954). (Russian)

N. G. Čebotarev [Mat. Sb. 34 (1927), 149–206] has generalized the concept of translation surfaces to surfaces admitting families of imprimitivity with respect to a transformation group $G$, that is, possessing families of curves which pass into each other under transformations of $G$. Such surfaces have either a continuous system of imprimitivity, or no more than four systems. Among them are surfaces of displacement in elliptic space, investigated by Bianchi [Lezioni di geometria differenziale, v. 2, parte 2, Zanichelli, Bologna, 1924] and by the author [Uč. Zap. Har'kov Gos. Univ. 28, Zap. Naučno-Issled. Inst. Mat. Meh. i Har'kov Mat. Obšč. (4) 20 (1950), 61–76; MR 14, 405]. In the present paper such surfaces are studied with the aid of their equation $X=Y(u)Z(v)$, where $Y$ and $Z$ are quaternions depending on one variable, and in particular those on which the system of imprimitivity is continuous. This leads to ruled surfaces with Clifford parallels as generators (cylindrical surfaces of elliptic space). The equations of the problem lead to three different cases, each of which is investigated in turn. *D. J. Struik.*

**Speranza, Francesco. Le trasformazioni puntuali fra spazi sovrapposti nei casi particolari.** Boll. Un. Mat. Ital. (3) 10 (1955), 513–521.

Una trasformazione puntuale $T$ tra due spazi lineari $S_r$ sovrapposti determina in una coppia regolare $A$, $\bar{A}$ di punti corrispondenti una omografia $\Omega$. Questa nota completa lo studio già fatto dall'autore nel caso che la $\Omega$ sia generale [lo stesso Boll. (3) 10 (1955), 61–68; MR 17, 78]: l'autore classifica, studia ed assegna riferimenti canonici delle trasformazioni puntuali nel caso che la $\Omega$ sia particolare. *C. Longo* (Roma).

**Pan, T. K. Torsion of a vector field.** Proc. Amer. Math. Soc. 7 (1956), 449–457.

In this paper the author has introduced the notion of indicatrix torsion which includes geodesic torsion as a special case. Classical theorems on geodesic torsion by Enneper and Bonnet are generalized. Directions on a surface for which the indicatrix torsion attains extreme values are called principal directions of torsion in analogy with corresponding notions involving curvature. These naturally lead to principal torsions, total torsion and mean torsion. *E. T. Davies* (Southampton).

**Saban, Giacomo. Sul teorema dei quattro vertici.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 19 (1955), 251–258.

Let $C$: $x=x(s)$ be a closed space curve with continuous curvature $\varrho^{-1}$ and torsion $\tau$. a) If $C$ is a convex spherical curve and $\sigma=(xx'x'')>0$, then there exist at least four points of $C$ with $\sigma'=(xx'x''')=0$; b) if the spherical representation of the binormal unit vectors is convex, then there exist at least four points of $C$ with $(\varrho/\tau)'=0$. Some mechanical interpretations of these results are also given. *L. A. Santaló* (Buenos Aires).

**Chung, Tong-Der. The projective differential geometry of certain pairs of plane curves.** J. Chinese Math. Soc. (N.S.) 2 (1953), 157–166. (Chinese. English summary)

Let two curves, $C$, $\bar{C}$ in a projective plane intersect at a common point 0 of inflexion with distinct tangent lines. A projective invariant $I$ is determined by the neighborhood of the fifth order of $C$, $\bar{C}$ at 0, and a simple geometric interpretation of $I$ is given by use of Bompiani's osculants. According as $I$ vanishes or not, four different types of the canonical power series expansions of $C$, $\bar{C}$ at 0 are obtained by a geometric determination of the triangle of reference and the unit point, and the projective invariants in the expansions of each type are interpreted geometrically in terms of certain cross ratios. Similar results are mentioned for two plane curves with a common tangent line at two points of inflexion. The method used

in this paper is the same as that in a paper of the reviewer [Bull. Amer. Math. Soc. **49** (1943), 786–792; MR **5**, 108].

*C. C. Hsiung* (Bethlehem, Pa.).

**Ansermet, A. Le calcul semi-graphique de la déformation de réseaux projetés dans un système conforme.** Schweiz. Z. Vermessg. Kulturtech. Photogr. **54** (1956), 228–233.

In the conformal representation of a geodetic survey net onto a map, it is useful to determine the geodesic curvature of the projected geodesic for the purpose of obtaining the angular corrections at the two endpoints of the line. Using the classical formulas for the change of geodesic curvature of a conformally mapped curve, the author derives approximation formulas for the sum, difference and quotient of these angular corrections, which prove amenable to graphical representation. It is claimed that such a graph can be used to compute these corrections, but due to the degree of accuracy required in geodetic work, it appears that their primary value would be for checking purposes.

The theory is developed for a general conformal mapping. The specific examples given and graphs displayed are for the case of the Swiss geodetic net.

*B. Chovitz* (Washington, D.C.).

**Picasso, Ettore. Sopra una generalizzazione della conica di Kommerell cui da luogo un sistema planare di curve su una superficie di $S_4$.** Boll. Un. Mat. Ital. (3) **11** (1956), 31–37.

Sia $F$ una superficie di $S_4$ (a punti non parabolici), per ogni punto $x$ della quale sia definito un piano (di appoggio) $\omega$ (per $x$). L'A. ha già dimostrato [Atti 1° Congresso Un. Mat. Ital., Firenze, 1937, Zanichelli, Bologna, 1938, pp. 346–349] che il luogo dei punti comuni ad $\omega$ e al piano uscente dal punto infinitamente vicino ad $x$ è una conica, che generalizza la nota conica di Kommerell. Nel presente lavoro si associa invece ad ogni punto $x$ di $F$ un sistema di quadriche — legate al piano $\omega$ e ai trasformati di Laplace di $x$ —: in relazione a tali quadriche si studia altresì una particolare connessione proiettiva sulla superficie.

*V. Dalla Volta* (Roma).

**Prvanovitch, Mileva. Une généralisation des espaces totalement géodésiques.** C. R. Acad. Sci. Paris **242** (1956), 2219–2221.

**Prvanovitch, Mileva. Propriétés des espaces parageódésiques.** C. R. Acad. Sci. Paris **242** (1956), 2500–2502.

In the first note the parageodesic curvature of a curve in a $V_n$ in $V_m$ in $V_l$ is defined by means of an arbitrary $(l-m)$-direction given in each point of $V_m$. The $V_n$ is said to be parageodesic in $V_m$ with respect to this set of $(n-m)$-directions if the enforced curvature vector of every curve of $V_n$ with respect to $V_l$ lies in the local $(l-m)$-direction. In the second note the parageodesic curvature tensor of valence 3 and the parageodesics of $V_n$ are defined. It is proved that if $V_n$ is parageodesic, every parageodesic curve in $V_n$ is also parageodesic in $V_m$.

*J. A. Schouten* (Epe).

**Bompiani, Enrico. Sull'equazione differenziale di Jacobi ed altre analoghe.** Ann. Mat. Pura Appl. (4) **39** (1955), 15–24.

In homogenen Koordinaten $x^i$, $dx^i$ und mit $\varrho y^i = a_k{}^i x^k$ schreibt sich die Jacobische Differentialgleichung

$$(a+a'x+a''y)(xdy-ydx)-(b+b'x+b''y)dy+(c+c'x+c''y)dx=0$$

kurz:

$$(*)\qquad |xdxy|=0.$$

Die Trägergerade des Linienelementes $E_1$ im Punkt $x(x^i)$ ist also die Verbindungsgerade der Punkte $(x)$ und $(y)$. Da die Projektivität

$$(**)\qquad x^i=(a_k{}^i-\tau\delta_k{}^i)x^k$$

($\delta_k{}^i=1$, $i=k$, $=0$, $i\neq k$, $\tau$ beliebig) die Gleichung (*) nicht ändert, kann man $\varrho y^i$ durch

$$\rho y^i=\alpha^i x^i$$

und (*) durch

$$(***)\qquad a_1\frac{dx^1}{x^1}+a_2\frac{dx^2}{x^2}+a_3\frac{dx^3}{x^3}=0$$

ersetzen ($a_1=\alpha_2-\alpha_3$, $a_2=\alpha_3-\alpha_1$, $a_3=\alpha_1-\alpha_2$, $a_1+a_2+a_3=0$). Die Integralkurven von (***) sind die Klein-Lieschen $W$-kurven. — Zu drei Linienelementen $E_1$ allgemeiner Lage in der projektiven Ebene gehört die Invariante

$$I=\frac{v_i x^i w_j y^j u_l z^l}{w_i x^i u_j y^j v_l z^l}$$

sobald $x(x^i)$, $y(y^i)$, $z(z^i)$ die Anfangspunkte der Elemente und $u(u_i)$, $v(v_i)$, $w(w_i)$ die Geradenkoordinaten der zugehörigen Trägergeraden bedeuten. Läuft der Punkt $z$ auf einer Kurve, so ändert sich $I$ im allgemeinen, bleibt jedoch konstant, wenn die Kurve Integralkurve der Differentialgleichung

$$I\frac{dx}{x}+\frac{dy}{y}=0 \text{ mit } \frac{a_1}{a_2}=I$$

ist ($dx^3=0$). Nach diesem Ergebnis betrachtet Verfasser die Zuordnung

$$\varrho y^i=a^i{}_{kl}x^k y^l$$

(deren Umkehrung im allgemeinen nicht mehr eindeutig ist) und die Differentialgleichung

$$|xdxy|=\begin{vmatrix} x^1, & x^2, & x^3 \\ dx^1, & dx^2, & dx^3 \\ a_{hk}{}^1 dx^h dx^k, & a_{hk}{}^2 dx^h dx^k, & a_{hk}{}^3 dx^h dx^k \end{vmatrix}=0.$$

Derartige Transformationen gibt es $\infty^{17}$; sie sind von allgemeinerem Charakter als die (ebenen) quadratischen Cremonatransformationen, deren es nur $\infty^{14}$ gibt. Von besonderer Bedeutung unter ihnen sind die quadratischen involutorischen Transformationen, aus welchen ein graphisches Integrationsverfahren für die Differentialgleichung gewonnen werden kann (das auch noch im nicht-involutorischen Fall durchführbar bleibt). Sodann werden die Kurven bestimmt, deren Linienelemente mit zwei (geordneten) Paaren weiterer Linienelemente projektive Invarianten von konstantem Quotienten bestimmen, ferner solche, deren Tangenten andere besondere Bedingungen auferlegt werden. Allgemein kann man $m$ (geordnete) Paare von Linienelementen in der projektiven Ebene vorgeben. Jedes dieser Elemente bestimmt eine projektive Invariante mit jedem dieser Paare. Gefragt wird nach solchen Linienelementen, daß die Linearkombination $\lambda_1 I_1+\lambda_2 I_2+\cdots+\lambda_m I_m$ der den $m$ Paaren entsprechenden Invarianten $I_1, I_2, \cdots, I_m$ mit bestimmten $\lambda_1, \lambda_2, \cdots, \lambda_m$ verschwindet. Diese Linienelemente erweisen sich als Integralkurven einer Differentialgleichung erster Ordnung und $m$-ten Grades.

*M. Pinl* (Köln).

**Hu, Hou-Sung. Some special affinely connected spaces.** Acta Math. Sinica 5 (1955), 325–332. (Chinese. English summary)

This is a continuation of a previous paper by the author [same Acta 3 (1953), 343–357; MR 17, 784] and studies affine connections which are conjugate with respect to an antisymmetric tensor. Two affine connections $\Gamma_{ij}^k$ and $\check{\Gamma}_{ij}^k$ are called common-pseudo, if there exists a vector $q_i$ such that $\Gamma_{ij}^k = \check{\Gamma}_{ij}^k + q_i \delta_j^k$. The following theorem is proved: In an $n$-space let $\underset{1}{\Gamma}, \cdots, \underset{n}{\Gamma}$ be $n$ affine connections conjugate with respect to an anti-symmetric tensor $e_{i_1 \cdots i_n}$. Then the following are true: 1) one of them, say $\underset{n}{\Gamma}$, can be chosen arbitrarily; 2) the remaining connections $\underset{1}{\Gamma}, \cdots, \underset{n-1}{\Gamma}$ are common-pseudo with $\underset{n}{\Gamma}$ (hence mutually common-pseudo); 3) the common-pseudo vectors $\underset{1}{q_k}, \cdots, \underset{n-1}{q_k}$ satisfy the condition

$$\underset{1}{q_k} + \underset{2}{q_k} + \cdots + \underset{n-1}{q_k} = \partial_k \log e - \underset{n}{\Gamma}_{ks}^s.$$

From this theorem several geometrical consequences are drawn. *S. Chern* (Chicago, Ill.).

**See also:** Kuiper and Yano, p. 5; Seminairés de H. Cartan, p. 69; Günther, p. 79.

### *Riemannian Geometry, Connections*

**Sumitomo, Takeshi. On a special class of Riemannian spaces.** Tensor (N.S.) 5 (1956), 201–204.

A Euclidean space is characterised by the existence of coordinate systems in which the Christoffel symbols vanish. This fact led the author to consider such $n$-dimensional Riemannian spaces in which there exist coordinate systems for which the Christoffel symbols are constants. For $n=2$ it is shown that the space must be (locally) Euclidean, but as the author himself points out, he did not succeed in deriving a form of the metric tensor for the general case.

*H. Rund* (Durban).

**Ôtsuki, Tominosuke. Note on the isometric imbedding of compact Riemannian manifolds in Euclidean spaces.** Math. J. Okayama Univ. 5 (1956), 95–102.

Some theorems are given on the isometric imbedding of compact Riemannian manifolds in Euclidean space, which extend results of Chern and Kuiper [Ann. of Math. (2) 56 (1952), 422–430; MR 14, 408] and the author [Proc. Japan Acad. 29 (1953), 99–100; MR 15, 647]. For any point $p$ of the manifold $M$ let $k(p)$ be the minimum number of linear differential forms in terms of which the curvature forms of $M$ at $p$ can be expressed, and let $k(M) = \max_{p \in M} k(p)$. It is proved that a compact locally non-Euclidean Riemannian manifold $M$ of dimension $n$ cannot be isometrically imbedded in an Euclidean space of dimension $2n - k(M)$. Next suppose the curvature tensor be of the form $R_{ijkh} = H_{ik}H_{jh} - H_{jh}H_{jk}$, where $H_{ik}$ is a symmetric tensor. Let $\varrho(p)$ be the absolute value of the signature of $H_{ik}$ at $p$, and let $\varrho(M) = \max_{p \in M} \varrho(p)$. Then such a manifold $M$, if compact, cannot be isometrically imbedded in an Euclidean space of dimension $2n - \frac{1}{2}(k(M) + \varrho(M))$. Finally, some theorems for the three-dimensional case are deduced. *S. Chern* (Chicago, Ill.).

**Wintner, Aurel. Remarks on binary Riemannian metrics.** Rend. Circ. Mat. Palermo (2) 5 (1956), 59–72.

This paper contains various remarks which center around the question of isothermal normal forms of a positive definite binary Riemannian metric. Several applications are made of the following theorem due to Chern, Hartman and Wintner [Comment. Math. Helv. 28 (1954), 301–309; MR 16, 622]: Every regular $C^1$-metric is $C^2$-isometric to an isothermic regular $C^1$-metric. These applications include theorems on the second fundamental form of an imbedded surface of positive Gaussian curvature and the asymptotic behavior of the eigenvalues and eigenfunctions of a certain partial differential equation of the second order on a closed orientable surface of genus zero, etc. Similar questions and theorems are given concerning the "transversal geodesic" normal form $dx^2 + G(x, y)dy^2$. *S. Chern* (Chicago, Ill.).

**Hu, Hou-Sung. On the deformation of a Riemannian metric $V_m$ of class I which preserves the mean curvature.** Acta Math. Sinica 6 (1956), 127–137. (Chinese. English summary)

The object of the present paper is to investigate the deformation of $Vm \subset Em+1$ which preserves the mean curvature. Of course, the metric must necessarily be of rank 2. In the theory of Yanenko the system of determination which takes the form

$$(S) \qquad \Gamma_{\alpha j}^i \lambda_{ik} - \Gamma_{\alpha k}^i \lambda_{ij} = 0 \ (\alpha = 3, \cdots, m;\ i, j = 1, 2),$$

in suitable coordinates plays an important rôle. In our case we have the following result: (i) When the system $(S)$ is of rank 2, the hypersurface is non-deformable. (ii) When $(S)$ is of rank 1, the hypersurfaces is either non-deformable or deformable discretely. (iii) When $(S)$ is of rank 0, the problem is reduced to the deformation of $V_2$ in $S_3$ (space of positive constant curvature) which preserves the mean curvature.

In order to obtain the final result of case (iii) we use, at first, Cartan's criterion for a system of exterior differential equations and thence derive that a pair of deformable surfaces with preservation of mean curvatures depends upon four arbitrary functions of a single argument. Next, using the method of T. Y. Thomas we give an explicit expression of the coefficients of the second fundamental form of $V_2 \subset S_3$ in terms of the metric tensor and the mean curvature in general. In consequence, there can also be obtained the expression of the coefficients of the second fundamental form of a general $V_m$ in the case (iii), those for the cases (i) (ii) being trivial. Thus we have generalized the results of T. Y. Thomas (1945) to the case $V_m \subset E_{m+1}$ $(m>2)$. *From the author's summary.*

**Aragnol, André. Connexions euclidiennes canoniquement associées à certaines structures presque-produit.** C. R. Acad. Sci. Paris 242 (1956), 339–341.

An almost-product structure over a manifold is determined by a direct sum decomposition $T = C + C'$ of the tangent space $T$ at every point. A Riemann metric is chosen so that $C$ and $C'$ are normal with respect to this metric. A connection in the bundle of planes $C$ ($C$-connection) is called euclidean if it induces a metric in the bundle of frames of $C'$. Theorem I states that there is a unique euclidean $C$-connection satisfying some natural conditions on the torsion. Theorem II says that if $C$ and $C'$ are normal there is a unique connection under which $C$ and $C'$ are invariant, and which satisfies the conditions of Theorem I. Other applications of Theorem I are

discussed, among which the almost-product structure of a fiber bundle in which a connection is given.

*A. Nijenhuis* (Seattle, Wash.).

**Čahtauri, A. I. On an invariant characteristic of a projectively deformable surface.** Soobšč. Akad. Nauk Gruzin. SSR **17** (1956), 3–6. (Russian)

For the *R*-surfaces of Fubini-Čech [Introduction à la géométrie projective différentielle des surfaces, Gauthier-Villars, Paris, 1931, pp. 85–86], which are non-ruled projectively deformable surfaces, the parameters of the asymptotic lines can be selected such that $\beta_v=\gamma_u$, where $\beta du^3+\gamma dv^3=0$ gives the Darboux directions. In the present paper it is shown that an invariant characteristic of such surfaces is the gradient character of the vector $r_i$, where

$$r_i=\frac{1}{12J}D^{kmn}\nabla_i D_{kmn},$$

$$D_{ijk}=\nabla_k b_{ij}-\tfrac{1}{2}(t_k b_{ij}+t_i b_{kj}+t_j b_{ij}),\quad J=\tfrac{1}{8}D^{ijk}D_{ijk},$$

$t_i=\frac{1}{2}\bar{b}^{ks}(2\nabla_k b_{is}-\nabla_i b_{ks})$, and $b_{ij}\,du^i du^j=0$ gives the asymptotic lines. Covariant differentiation is defined with respect to a basic set of four points $x^\alpha$, $X^\alpha$, $\partial_1 x^\alpha$, $\partial_2 x^\alpha$, $\alpha=1$, 2, 3, 4; $x^\alpha$ on the surface $x^\alpha=x^\alpha(u^1, u^2)$, $X^\alpha$ not in the tangent plane; and such that

$$\partial_{ij}x^\alpha=\Gamma_{ij}{}^m\partial_m x^\alpha+p_{ij}x^\alpha+b_{ij}X^\alpha;$$

the $\Gamma_{ij}{}^m$ are the coefficients of connection, $p_{ij}$ is arbitrary. The directions of Darboux are given by $D_{ijk}du^i du^j du^k=0$; raising of indices is with the aid of $\bar{b}^{ij}$, the inverse of $b_{ij}$.

*D. J. Struik* (Cambridge, Mass.).

**Gu, Čao-Hao. On parallel connections of plane elements and nonholonomic manifolds.** Acta Math. Sinica **5** (1955), 383–392. (Chinese. Russian summary)

Let $A_n$ be a symmetric affine connection of dimension $n$. The theorems in this paper include the following: 1) For $n>2$ and $1\leqq m\leqq n-1$ suppose that to every $m$-dimensional plane element there exists a field of plane elements absolutely parallel to it. Then $A_n$ is flat. [For $m=1$ cf. Eisenhart, Non-Riemannian geometry, Amer. Math. Soc. Colloq. Publ., v. 8, New York, 1927]. 2) If $A_n$ admits a non-holonomic manifold defined by $F_{\sigma i}dx^i=0$, then the vectors $V^i$ satisfying $F_{\sigma i}V^i=0$ remain on the non-holonomic manifold when parallelly displaced. 3) If a totally geodesic non-holonomic manifold is admitted by $A_n$, it is integrable.

*S. Chern* (Chicago, Ill.).

**Mutō, Yosio. On some properties of a kind of affinely connected manifolds admitting a group of affine motions. I.** Tensor (N. S.) **5** (1955), 127–142.

In a previous note [Tensor (N.S.) **5** (1955), 39–53; MR **17**, 407] the author has given the forms of the curvature tensor of an $A_n$, $n\geqq 5$, admitting a group of affine motions of order $r>n^2-n$ and for an $A_n$, $n\geqq 7$ admitting such a group of order $r>n^2-2n$ (there is a misprint in the synopsis). In the present paper the necessary and sufficient conditions for the admission of the groups are pursued more precisely. It is found that in the first case mentioned above the space is projectively flat and that in the second case the form of the curvature tensor can be simplified for $n\geqq 7$ into

$$R^{\lambda}_{\cdot\mu\nu\omega}=A^\lambda B_{\mu\nu\omega}+\delta^\lambda{}_\mu(P_{\nu\omega}-P_{\omega\nu})+\delta_\nu{}^\lambda P_{\mu\omega}-\delta_\omega{}^\lambda P_{\mu\nu}$$

and for $n=7$ into

$$R^{\lambda}_{\cdot\mu\nu\omega}=(A^\lambda P_\mu+C^\lambda Q_\mu)(P_\nu Q_\omega-P_\omega Q_\nu)+\delta^\lambda{}_\mu(P_{\nu\omega}-P_{\omega\nu})+\delta_\nu{}^\lambda P_{\mu\omega}-\delta_\omega{}^\lambda P_{\mu\nu}.$$

*J. A. Schouten* (Epe).

**Mocanu, P. Espaces partiellement projectifs.** Acad. R. P. Romîne. Stud. Cerc. Mat. **6** (1955), 495–528. (Romanian. Russian and French summaries)

A partially projective space $P_n{}^{n-k}$ is a space with symmetric affine connection $(A_n)$ which has the property that its autoparallel lines given by

$$\text{(1)}\qquad \frac{d^2x^i}{dt^2}=\Gamma^i_{jk}\frac{dx^j}{dt}\cdot\frac{dx^k}{dt}$$

lie in linear varieties of $k$ dimensions, when expressed in a suitable coordinate system. The superscript $n-k$ indicates the number of equations of the autoparallel lines, which are linear in the coordinates $x^1, x^2, \cdots, x^n$. If the linear varieties pass through a fixed point the $P_n{}^{n-k}$ is a "Kagan" space, otherwise it is a "general" $(n-k)$ fold projective space.

In this paper partially projective spaces are studied with the help of the general integral of (1) expressed as a Taylor series. The author obtains in a simple way Weyl's formulae for the Γ's of a projectively flat $A_n$ (=a general $P_n{}^{n-1}$) as well as Kagan's conditions for a subprojective space (=$P_n{}^{n-2}$ Kagan space) and Vranceanu's conditions for a general $P_n{}^{n-2}$. In this latter case a number of invariants with respect to projective transformations of the affine connection are derived. The vanishing of one set of these projective invariants provides the necessary and sufficient condition for the autoparallel lines (which are plane curves) to be parabolas.

By the same method the author investigates the case $k=3$ and then the general case. He gives necessary and sufficient conditions for an $A_n$ to be 1) a Kagan $P_n{}^{n-k}$; 2) a general $P_n{}^{n-k}$. These conditions are a system of partial differential equations in the Γ's ,of order $k-2$ in the first case, and of order k−1 in the second case. To these there have to be added a number of inequalities which express the condition that the $P_n{}^{n-k}$ is not a $P_n{}^{n-k+1}$.

*R. Blum* (Saskatoon, Sask.).

**Aragnol, André. Champ d'holonomie et sous-algèbre d'holonomie.** C. R. Acad. Sci. Paris **242** (1956), 1117–1120.

Let $E$ be a principal fiber bundle with group $G$, in which a connection is given. A field $C$ of vertical $p$-planes over $E$ is called stable if (1) under the natural correspondence between each vertical tangent plane and the Lie algebra $A(G)$, $C_x$ for each $x\in E$ represents a subalgebra of $A(G)$; (2) if $x'=xs$; $s\in G$, then $C_{x'}=\mathrm{ad}(s)C_x$; (3) the set $L(C)$ of semi-basic differential forms with values in $C$ satisfies $DL(C)\in L(C)$. (A form $\varphi$ is semi-basic if $\varphi(u_1, \cdots, u_p)=0$ whenever one of the $u$'s is a vertical vector.) A field $C$ of vertical $p$-planes over $E$ is called integrable if the field of subspaces $C+H$, where $H_x$ is the horizontal plane of the connection at $x$, is integrable.

Theorem I states that a stable field $C$ is integrable if and only if the curvature form $\Omega$ satisfies $\Omega\in L(C)$. Theorem II states that the special field $C(\Omega)$, arising from taking $\Omega(u, v)$ ($u$, $v$ tangent vectors to $E$) by any horizontal path from any point to any other, is stable and integrable, and that if $C$ is any other stable and integrable field, then $C(\Omega)\subset C$. $C(\Omega)$ is called the "field of holonomy", and $L(C(\Omega))$ the "subalgebra of holonomy". The theorem of Ambrose and Singer [Trans. Amer. Math. Soc. **75** (1953), 428–443; MR **16**, 172] easily follows from Th. II.

*A. Nijenhuis* (Seattle, Wash.).

**Haimovici, Adolf. Espaces à connexion affine qui admettent la notion d'aire.** Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. **6** (1955), 123–133. (Romanian. Russian and French summaries)

In a space with affine connection one can attach to two vectors $X^i$ and $Y^i$ an invariant which has the character of an area $(S)$, by the formula:

$$S^2=a_{ijhk}(X^iY^j-X^jY^i)(X^hY^k-X^kY^h),$$

where $a_{ijhk}=a_{hkij}=-a_{jihk}$ is the "areolar tensor". The condition that $S$ should be invariant under a parallel transport of $X^i$ and $Y^j$ leads to a system of linear partial differential equations of the first order in $a_{ijhk}$.

In the present paper two particular cases for $a_{ijhk}$ are discussed: 1) $a_{ijhk}=g_{ij}g_{hk}$ where $g_{ij}$ is skew symmetric in $i, j$. For $n=2$ the space is equiaffine, a result found earlier by the author. For $n>2$ conditions for the $g_{ij}$ are established. It is then shown how the coefficients of the affine connection can be determined. The area of a surface region depends, in this case, only upon the bounding curve. 2) There exists a coordinate system in which $g_{ijhk}\neq 0$ $(i=h, j=k)$ and $g_{ijhk}=0$ (otherwise). It is shown that, in this case, the space is Riemannian. *R. Blum.*

## *Algebraic Geometry*

Have

★**Conforto, Fabio. Abelsche Funktionen und algebraische Geometrie. Aus dem Nachlass bearbeitet und herausgegeben von W. Gröbner, A. Andreotti und M. Rosati.** Springer-Verlag, Berlin-Göttingen-Heidelberg, 1956. xi+276 pp. DM 41.80.

The content of this book is based on lecture notes (1940–51) of the late F. Conforto. The material deals with the classic, essentially analytic, theory of abelian functions of arbitrary genus. In general the methods used are those of the French and German masters, which are supplemented and deepened by the results of the theory of several complex variables as may be found in the book of Behnke-Thullen. The authors state explicitly in the introduction that their aim is to present this analytic theory which is "not to be encumbered too much by modern abstractions". This program is indeed carried out and the reader is thus lead to about the stage of the development of the theory of abelian functions where the work of authors like Albert, Scorza and H. Weyl on Riemann matrices, Chevalley on Lie groups, and Weil on abelian varieties begins. Essentially the following three topics are covered: (i) Defining properties of a Riemann matrix as a period matrix of meromorphic functions of a finite number of variables, sufficiency of these conditions for the existence of abelian functions using the representation of intermediary functions as linear combinations of $\vartheta$-series; (ii) existence of a projective model without singularities for the field of all abelian functions belonging to a given Riemann matrix (proof of C. L. Siegel) with a discussion of Picard varieties, also algebraic correspondences on such, and some very brief indications on reducible Riemann matrices (fiberings etc. are not mentioned), discussion of the model without singularities as a group variety, and finally and interpretation and consequences of the theorem of Appel-Humbert on intermediary functions in terms of ideal theory leading to algebraic systems of highest dimensional varieties on Picard varieties; a discussion of Wirtinger and Kummer manifolds as examples of abelian varieties of rank 2; and (iii) a discussion of algebraic correspondences between Picard varieties and the associated relations of Hurwitz for the corresponding Riemann matrices; emphasis is placed on the formal analogy to Hurwitz' ideas for algebraic curves and thus some significant results from the theory of functions of several variables are needed (see the footnote on pp. 223–4). The reader who is not acquainted with some of the basic tools of modern algebraic geometry may be advised to simplify his studies by preparing a careful record of the various meanings of the term "general" as used by the authors, and also the necessary techniques pertaining to same. This book is certainly useful as a record of the analytical theory of abelian functions.

*O. F. G. Schilling* (Chicago, Ill.).

**Bompiani, E. Sulla varietà rappresentativa degli elementi lineari del piano proiettivo.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **19** (1955), 207–212.

The set of elements formed by points $(x^1, x^2, x^3)$ plus straight lines $(u_1, u_2, u_3)$ of the projective plane may be represented by the points $\xi_j^i=x^iu_j$ of a variety $V_4^6$ in $S_8$. The set of linear elements $E_1$ formed by points plus straight lines which intersect, will be represented by the $V_3^6$ (in $S_7$) intersection of $V_4^6$ with the hyperplane $\xi_i^i=0$ [F. Severi, Ann. Mat. Pura Appl. (4) **19** (1940), 153–242; MR **7**, 476; G. Scorza, **15** (1908), 217–273]. The author gives a parametric representation of $V_3^6$ and new properties of this variety. Let $S_3^*$ be the tangent 3-plane to $V_3^6$ at the point corresponding to the element $x^1=0$, $x^2=0$, $x^3=1$, $u_1=0$, $u_2=1$, $u_3=0$; the tangent 3-plane at any other point $P$ of $V_3^6$ always intersect $S_3^*$ in a point $P^*$. The map $P\to P^*$ of $V_3^6$ on $S_3^*$ is investigated in detail. *L. A. Santaló* (Buenos Aires).

**Bompiani, Enrico. Ancora sulla varietà rappresentativa degli elementi lineari del piano proiettivo.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **19** (1955), 361–367 (1956).

Following the work of the preceding review, the author proves that a conic of the projective plane considered as locus of its tangent linear elements $E_1$ is represented on the variety $V_3^6$ by a rational normal quartic $C^4$. To the set of elements $E_1$ whose base points are on a nondegenerate conic $C$ and the straight lines pass through a fixed point $P$ of $C$ correspond a skew cubic $C^3$ on $V_3^6$. If $P$ does not belong to $C$, the image is a quartic $Q^4$. Some properties of these $C^3$, $C^4$, $Q^4$ and certain related ruled surfaces are given. *L. A. Santaló* (Buenos Aires).

★**Spampinato, Nicolò. Lezioni di geometria superiore. Vol. VII. Elementi di geometria algebrica e corpogeometria sopra una varietà. Le superficie di Riemann e gl'integrali abeliani.** Casa Editrice Raffaele Pironti e Figli, Napoli, 1950. 616 pp. 3500 Lire.

★**Spampinato, Nicolò. Lezioni di geometria superiore. Vol. VIII. Enti iperalgebrici e geometrie fondamentali nell' $S_2$ complesso. Rappresentazioni complesse delle corpoproiettività dell' $S_1$ biduale. Derivazione ed integrazione nel campo ipercomplesso.** Casa Editrice Raffaele Pironti e Figli, Napoli, 1952. 676 pp. 3500 Lire.

**Spampinato, Nicolò. Inviluppo e congruenza delle tangenti ai rami di una falda bidimensionale piana o tridimensionale dell' $S_3$.** Rend. Accad. Sci. Fis. Mat. Napoli (4) **22** (1955), 62–66 (1956).

**Spampinato, Nicolò. Prolungamento di una ipersuperficie dell' $S_r$ complesso nel campo quadripotenziale.** Rend. Accad. Sci. Fis. Mat. Napoli (4) **22** (1955), 67–74 (1956).

**Spampinato, Nicolò. Sulle condizioni di razionalità per una superficie tripotenziale.** Rend. Accad. Sci. Fis. Mat. Napoli (4) **22** (1955), 75–93 (1956).

**Spampinato, Nicolò. Carattere singolare e carattere cuspidale di una curva algebrica completa.** Rend. Accad. Sci. Fis. Mat. Napoli (4) **22** (1955), 123–143 (1956).

**Spampinato, Nicolò. Le falde tridimensionali dell' $S_5$ determinate dai rami lineari e superlineari di una curva algebrica completa.** Rend. Accad. Sci. Fis. Mat. Napoli (4) **22** (1955), 202–220 (1956).

**Spampinato, Nicolò. Rappresentazione in $S_5$ del piano complesso completo e relativo gruppo di trasformazioni birazionali.** Rend. Accad. Sci. Fis. Mat. Napoli (4) **22** (1955), 249–261 (1956).

**Fadini, Angelo. Le superficie iperellittiche dell' $S_3$ triduale e la loro rappresentazione complessa.** Rend. Accad. Sci. Fis. Mat. Napoli (4) **22** (1955), 154–160 (1956).

**Washnitzer, G. The characteristic classes of an algebraic fiber bundle. I.** Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 433–436.

In this note, the author gives a purely algebro-geometric definition of the characteristic classes of an algebraic fiber bundle with a vector space for fiber. The characteristic classes turn out to be rational equivalence classes on the base variety which is supposed to be free from multiple points and to admit a projective model, that is, the base variety is in biregular correspondence with a subvariety of some projective space. *S. T. Hu.*

**Severi, Francesco. Fondamenti per la geometria sulle varietà algebriche. III.** Ann. Mat. Pura Appl. (4) **41** (1956), 161–199.

[For parts I–II see Rend. Circ. Mat. Palermo **28** (1909), 33–87; Ann. Mat. Pura Appl. (4) **32** (1951), 1–81; MR **13**, 979.] The main results of this paper have been announced previously, [C. R. Acad. Sci. Paris **242** (1956), 59–61, 225–227; MR **17**, 664] and are described in the reviews just quoted. To these the reviewer would add the following remarks.

The author apparently claims to prove by algebro-geometric reasoning the equivalence of two definitions of $k$-ple integrals of the second kind on an algebraic variety, proposed by Hodge and Atiyah [Ann. of Math. (2) **62** (1955), 56–91; MR **17**, 533]. Denoting a $k$-ple differential form by $\omega$ these definitions are, in Severi's notation, (a'): $\int\omega$ is of the second kind on $M_r$ if there is a hypersurface $W_{r-1}$ on $M_r$, containing the singularities of $\omega$, such that $\int\omega$ vanishes over any $k$-cycle of $M_r - W_{r-1}$ which bounds in $M_r$; (b'): $\int\omega$ is q the second kind if, for each point $P$ of $M_r$ there is a $(k-1)$-form $\theta$ such that $\omega - d\theta$ is holomorphic in the neighbourhood of $P$. Atiyah and Hodge proved that (subject to a certain conjecture concerning the singularities of the singular variety of $\omega$), (a') implies (b'). The author does not seem to consider this problem, but outlines a proof {the details of which are not clear to the reviewer} that (b') implies (a').

The latter part of this paper contains some interesting suggestions of problems awaiting solution. *J. A. Todd.*

**Severi, Francesco. Contributi alla teoria delle irregolarità d'una varietà algebrica.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 7–16.

The main theme of this paper is the relationship between the irregularities of a variety $V_r$ [cf. Severi, C. R. Acad. Sci. Paris **242** (1956), 59–61, 225–227; MR **17**, 664] and the existence of $\infty^h$ involutions of systems of $V_{r-h}$ on $V_r$. The principal tool used is the consideration of the $h$-fold differential forms which are the product of $h$ simple differential forms, and especially the application of this idea to Picard varieties. Many results are given which could hardly be summarised in a review significantly shorter than the paper. *D. B. Scott* (London).

**Stoll, Wilhelm. Über meromorphe Modifikationen. V. Die Erzeugung analytischer und meromorpher Modifikationen durch $\sigma$-Prozesse.** Math. Ann. **130** (1955), 272–316.

On achève dans cet article la démonstration de résultats indiqués et préparés dans les articles précédents [Math. Z. **61**, 206–234 (1954), 467–488 (1955); **62** (1955), 189–210; Math. Ann. **130** (1955), 147–182; MR **16**, 689, 813; **17**, 530]. Il s'agit toujours des modifications $\mathfrak{M} = \mathfrak{M}[G, A, M, \tau; H, B, N, v]$, d'une variété complexe $G = A + M$, de dimension complexe 2 en une variété $H = B + N$ de même dimension, les ensembles singuliers $M$ et $N$ étant des sous ensembles d'un ensemble analytique de dim complexe 1 au plus. On donne un procédé constructif permettant de réaliser un ensemble de modifications ponctuelles $\sigma(P)$, $\sigma(P)$ étant un éclatement de Hopf ou substitution au point $P$ de l'espace projectif des directions complexes issues de $P$; l'ordre dans lequel les modifications $\sigma(P)$ sont faites n'est pas indiffèrent comme le montre un exemple donné au début du mémoire; on ordonne les $\sigma(P)$ au moyen d'un ensemble partiellement ordonné d'indices (arbre) $p$, $p_1$ et $p_2$ ayant un antécédent commun $q = p_1 \cap p_2$, et on définit des ensembles d'opérateurs correspondants conduisant à une variété limite; le procédé qui s'accompagne de nombreux détails et de précisions descriptives (certains énoncés dépassent une page) constitue une contribution intéressante à la construction d'espaces obtenus comme limites de systèmes, au sens de Freudenthal. On établit alors que si $\mathfrak{M}$ est une modification analytique, au sens précisé dans I, elle est réalisable, à une équivalence analytique près, par un arbre de modifications $\sigma$ (la possibilité de réaliser $\mathfrak{M}$ avec des modifications $\sigma$ avait été établie par H. Hopf quand $M$ est réduit à un point). On établit ensuite le même résultat quand il s'agit d'une modification $\mathfrak{M}$ méromorphe, $\mathfrak{M}$ et $\mathfrak{M}^{-1}$ étant ouvertes, au sens précisé par l'auteur dans I; la démonstration, qui donne en même temps la construction de l'arbre, repose sur les résultats de IV. Une étude est faite ensuite pour préciser les simplifications propres au cas où $G$ et $H$ ont une base dénombrable des ouverts; on pourra alors réaliser $\mathfrak{M}$, supposée toujours méromorphe et ouverte ainsi que $\mathfrak{M}^{-1}$, en opérant une suite de modifications, chacune consistant à faire éclater les points d'un ensemble $M_2$ sans points d'accumulation. *P. Lelong.*

★**Séminaires de H. Cartan, 1953-1954. Chapters XVI-XIX and Séminaire Bourbaki.** Mathematics Department, Massachusetts Institute of Technology, Cambridge, Mass., 1955. **44** pp. **$1.25.**

The first two chapters of this excerpt from the Cartan

Seminar of 1953–1954 are primarily devoted to the developing of the methods used in Cartan and Serre, C. R. Acad. Sci. Paris **237** (1953), 128–130 [MR **16**, 517]. The first section is titled Deux théorèmes sur les applications complètement continues, and discusses in detail two theorems of L. Schwartz [ibid. **236** (1953), 2472–2473; MR **15**, 233].

We shall let $E$ and $F$ denote two Fréchet spaces; that is, locally convex complete linear spaces [Bourbaki, Elements de mathématique, XV, Actualités Sci. Ind., no. 1189, Hermann, Paris, 1953; MR **14**, 880]. A linear transformation $v: E \to F$ is completely continuous if for some open neighborhood $U$ of the origin, the closure of $v(U)$ is compact. The first theorem of L. Schwartz asserts that if $u$ and $v$ are two continuous linear functions of $E$ into $F$, and if $v$ is completely continuous, while $u$ is an isomorphism onto $u(E)$ and $u(E)$ is closed, then the kernel of $w=u+v$ is finite dimensional, while the image $w(E)$ is closed. The next theorem, which may be derived from the first asserts that if $u$ and $v$ are linear continuous functions of $E$ into $F$ and if $u$ is onto, while $v$ is completely continuous, then $w=u+v$ is a homomorphism onto a closed subspace $w(E)$ which has finite codimension. These two results are discussed and derived in chapter XVI.

The following section, chapter XVII, is titled Un théorème de finitude. The theorem proved here is about the cohomology groups of a compact analytic manifold with coefficients in a coherent sheaf. [For the notions of cohomology with coefficients in a sheaf we may refer for example to the H. Cartan Seminar of 1950–51 [MR **14**, 670] or to H. Cartan, Colloque sur les fonctions de plusieurs variables, [Bruxelles, 1953, Thone, Liège, 1953; MR **16**, 235].] The theorem in question is: Suppose $V$ is a compact, complex analytic manifold. Let $F$ be a coherent analytic sheaf on $V$. Then the cohomology groups $H^n(V, F)$ are all finite dimensional. This was also shown by K. Kodaira [Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 865–868; MR **16**, 74]. Use is made of the two theorems of L. Schwartz. Also, the theory of Stein manifolds is employed. It is possible to cover the manifold $V$ by finitely many open sets, the intersection of these open sets being Stein manifolds. This is a covering which is acyclic with respect to coefficients in $F$. Using a method of A. Weil [Comment. Math. Helv. **26** (1952), 119–145; MR **14**, 307], the cohomology group $H^n(X; F)$ is isomorphic to the cohomology group $H^n(N(U); F)$, where $N(U)$ is the nerve of an open covering by Stein manifolds as previously described. When this is established, the results of Schwartz are applied to obtain the theorem.

The third section is entitled: Faisceaux Analytiques sur l'espace projectif. This begins with a discussion of the $d''$-cohomology of a complex analytic manifold. Thus, if $w$ is a form of type $(p, q)$, then $dw$ is the sum of a form of type $(p+1, q)$ and a form of type $(p, q+1)$, and $d''w$ is that form of type $(p, q+1)$. The first proposition is that a polycylinder (complex) is acyclic with respect to the $d''$-cohomology. From this, a Poincaré lemma is obtained for the operator $d''$ and the theorem of Dolbeault [C. R. Acad. Sci. Paris **236** (1953), 175–177; MR **14**, 673] is given. The remainder of this section, and also the next chapter, XIX, are devoted to studying coherent sheaves on projective space. The principal result is the complex analytic form of a theorem of Serre used in Faisceaux algébriques cohérents [Ann. of Math. (2) **61** (1955), 197–278; MR **16**, 953]. Also a proof of Chow's theorems to the effect that every complex analytic submanifold of complex projective space is algebraic is given.

The final section by J. P. Serre, is taken from the Bourbaki Seminar (March 1954) and is entitled Faisceaux analytiques. In this section complex analytic vector bundles are discussed, and the Serre duality theorem for such vector bundles is given. Thus if $V$ is a complex analytic manifold and if $B$ is a complex vector bundle on $V$, then let $A_B^{p,q}$ denote the differential forms on $V$ of type $(p, q)$ with coefficient in $B$. The space $A_B^{p,q}$ is made into a Frechet space with the Schwartz topology. If the two operators

$$A_B^{p,q-1} \xrightarrow{d''} A_B^{p,q} \xrightarrow{d''} A_B^{p,q+1}$$

are homomorphisms, then the vector spaces

$$H^q(V; \Omega_B^p) \text{ and } H^{n-q}\ (V\Omega_{B^*}^{n-p})$$

are dual, where $\Omega_B^p$ is the sheaf of germs of holomorphic $p$ forms on $V$ with coefficients in $B$, and $\Omega_{B^*}^{n-p}$ is the sheaf of germs of holomorphic $n-p$ forms on $V$ with coefficients in $B^*$ the bundle dual to $B$. There is also more discussion of coherent sheaves and some applications such as a proof of a lemma of Enriques-Severi is given.

These chapters contain a good deal of the essential material used in treating complex analytic manifolds. A large portion is devoted to algebraic geometry. There has since been quite a bit of interest in linear bundles and the problem of classifying them.

*P. E. Conner.*

**Segre, Beniamino. Sui punti fissi delle trasformazioni analitiche. I, II, III.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **19**, 200–204 (1955), 357–361 (1956); **20** (1956), 3–7.

An analytic self-transformation $T$ of an $n$-dimensional complex manifold $V$, which is reversible in the neighbourhood of a fixed point $O$ and induces a general homography in the space of tangent lines at $O$, can be expressed in the neighbourhood of $O$ is the form

$$u=ax+\sum a_{ij\cdots l}x^iy^j\cdots z^l$$
$$v=bx+\sum b_{ij\cdots l}x^iy^j\cdots z^l$$
$$\cdot \quad \cdot \quad \cdot$$
$$w=cx+\sum c_{ij\cdots l}x^iy^j\cdots z^l$$

where $(x, y, \cdots, z)$ and $(u, v, \cdots, w)$ are local coordinates of corresponding points, the coefficients $a$, $a_{ij\cdots l}$ etc. are complex numbers, and the summation is over non-negative integers $i, j, \cdots, l$ such that $i+j+\cdots+l\geqq 2$.

The main problem is whether, by a suitable transformation of local coordinates, the transformation can be linearised. This is restated in more general terms as a problem on the analytical equivalence of two different self-transformations $T$, $T'$ of different varieties $V$, $V'$ in the neighbourhoods of fixed points $O$, $O'$. It is a necessary condition for such equivalence that the "coefficients of dilatation" $a, b, \cdots, c$ and $a', b', \cdots, c'$ should be the same. If these coefficients are general (in a sense the author defines), or if they lie all inside or all outside the unit-circle, then this condition is also sufficient for formal equivalence in which $T$ and $T'$ are related by formal power series transformations in which questions of convergence are disregarded. It is shown however that this formal equivalence need not imply equivalence by complex-analytic (convergent) transformations.

*D. B. Scott* (London)

Have

★ **Longo, Carmelo. Approssimazione cremoniana delle trasformazioni puntuali fra piani in una coppia a jacobiano nullo di caratteristica zero.** Atti del Quarto Congresso dell' Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 374–385. Casa Editrice Perrella, Roma, 1953.

Una trasformazione cremoniana $T_C$ tra due piani $\pi$ e $\pi$ abbia il punto 0′ come fondamentale e sia 0 un punto della curva fondamentale $C$ corrispondente ad 0′.

Se nella coppia (0, 0′) la $T$ a jacobiano di rango zero la $C$ è riducibile: la conoscenza dello spezzamento della $C$ [Franchetta, Boll. Un. Mat. Ital. (2) **2** (1940), 332–341; MR **2**, 137], permette di determinare i punti 0 di $C$ sui quali la $T$ a jacobiano di range zero, e di determinarne il relativo sviluppo locale.

Da ciò si perviene alla caratterizzazione geometrica delle trasformazioni puntuali approssimabili mediante trasformazioni cremoniane nell'intorno di una coppia a jacobiano di rango zero. *E. Bompiani* (Roma).

# NUMERICAL ANALYSIS

## *Numerical Methods*

★ **Taussky, Olga. Some computational problems in algebraic number theory.** Proceedings of Symposia in Applied Mathematics. Vol. VI. Numerical analysis, pp. 187–193. Published by McGraw-Hill Book Company, Inc., New York, 1956 for the American Mathematical Society, Providence, R. I. $9.75.

The author summarizes previous work in computation, (mostly by hand) of integral bases, class structures, class numbers, and units. There is a gap between such work and theoretical (class-field theoretic) work. As an example of the benefits of the interplay of the two endeavors, she shows the field generated by $\theta^3-91\theta+273=0$ has class number three (without computation of units).

*Harvey Cohn* (St. Louis, Mo.).

**Ostrowski, Alexander. Verfahren von Steffensen und Householder zur Konvergenzverbesserung von Iterationen.** Z. Angew. Math. Phys. **7** (1956), 218–229.

If $\varphi_1(\zeta)=\varphi_2(\zeta)=\zeta$, then also $\Phi(\zeta)=\zeta$, where

$$\Phi(y)=\{y\varphi_1[\varphi_2(y)]-\varphi_1(y)\varphi_2(y)\}/\{y-\varphi_1(y)-\varphi_2(y)+\varphi_1[\varphi_2(y)]\}.$$

For the three iterations defined by $x_{t+1}=\varphi_1(x_t)$, $y_{t+1}=\varphi_2(y_t)$, and $Z_{t+1}=\Phi(Z_t)$, for $x_0$, $y_0$ and $Z_0$ in a suitable neighborhood of $\zeta$, it is known that if

$$[\varphi_1'(\zeta)-1][\varphi_2'(\zeta)-1]\neq 0,$$

then the third sequence converges to $\zeta$, and, further, that if the first two sequences converge, the third converges more rapidly. This was shown by Steffensen [Skand. Aktuarietidskr. **16** (1933), 64–72] for the case $\varphi_1=\varphi_2$, and by the reviewer [Householder, Principles of numerical McGraw-Hill, New York, 1953; MR **15**, 470] when $\varphi_1=\varphi_2$. The possibility of taking $\varphi_1\neq\varphi_2$ was first suggested by Hartree [Proc. Cambridge Philos. Soc. **45** (1949), 230–236; MR **10**, 574]. The author examines in general the rate of convergence of the third sequence and, further, asks the following question: Given $\varphi_1$ and $\varphi_2\neq\varphi_1$, under what conditions would the function $\Phi$ as defined above be preferred over either of the Steffenson functions using either $\varphi_1$ throughout or using $\varphi_2$ throughout. A necessary and sufficient condition is found which includes the requirement that $\varphi_2'(\zeta)=0\neq\varphi_1'(\zeta)$. A concluding paragraph distinguishes the Steffenson procedure from the Aitken $\delta^2$-process in that the latter was devised for a sequence $a_0, a_1, a_2, \cdots$ for which the sequence of quotients $a_{\nu+1}/a_\nu$ has a limit in which one is interested. Thus for computing the limit $\lambda$ one uses

$$(a_{\nu+1}a_{\nu-1}-a_\nu^2)/(a_{\nu+1}-2a_\nu+a_{\nu-1})$$

in place of $a_{\nu+1}/a_\nu$. *A. S. Householder.*

★ **Rosenbloom, P. C. The method of steepest descent.** Proceedings of Symposia in Applied Mathematics. Vol. VI. Numerical analysis, pp. 127–176. Published by McGraw-Hill Book Company, Inc., New York, 1956 for the American Mathematical Society, Providence, R. I. $9.75.

Aufgaben, die sich als Extremalaufgaben $f(y)=\min$; $y\in\mathfrak{H}$, einem Hilbertraum, formulieren lassen, können oft dadurch gelöst werden, daß man kontinuierlich in jedem Punkt $y$ in der Richtung fortschreitet, in der der Gradient $\nabla f$ am größten ist. Durch diese Methode, die dem Übergang von dem klassischen Dirichlet-Problem zur Wärmeleitungsgleichung entspricht und deren finite Approximation zu den wichtigsten Methoden der numerischen Analysis gehört, entsteht eine Differentialgleichung in $\mathfrak{H}$, $dy/dt=-\nabla f(y)$ deren Lösungsexistenz für alle $t>0$ zusammen mit Fehlerabschätzungen für $\|y(t)-y_\infty\|$ und $f(y(t))-f(y_\infty)$ hier untersucht wird. Im Falle eines quadratischen Variationsproblems $f(y)=\frac{1}{2}(By, y)-(z, y)$ ($B\geqq 0$ nicht notwendig beschränkt) entsteht $y'(t)=-By+z$, deren Lösung durch Spektralzerlegung von $B$ beschrieben wird oder durch eine „Operator-Parametrix-Methode" gefunden werden kann: $V(t)=U'(t)+BU(t)$ sei beschränkt,

$$\eta_0(t)=z-V(t)y_0;\ \eta(t)=\eta_0(t)-\int_0^t V(t-\tau)\eta(\tau)d\tau$$

unter gewissen Bedingungen ergibt sich die Lösung

$$y(t)=U(t)y_0+\int_0^t U(t-\tau)\eta(\tau)d\tau.$$

Auch die „quasilineare" Gleichung $y'(t)=-B(y)y+F(y)$, auf die sich das allgemeine Problem zurückführen läßt, sowie das Eigenwertproblem für einen symmetrischen Operator (als Extremalproblem mit Nebenbedingung) wird behandelt. Einfache Beispiele aus der Variationsrechnung erläutern die allgemeinen Sätze.

*D. Morgenstern* (Berlin).

Have

★ **Collatz, L. Numerische und graphische Methoden.** Handbuch der Physik. Bd. II., pp. 349–470. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1955. DM 88.00.

As the name implies this is a handbook designed to compress the maximum amount of information about numerical methods into the least possible space (122 pages). The author succeeds quite well in carrying out this difficult assignment. He does so by careful organization, by a terse treatment of each topic, and by omission of proofs and lengthy explanations.

The following synopsis indicates the range of topics selected.

A. General aids for computers, covers calculation with rounded numbers, slide rules, nomograms of half a dozen

different types, adjustment and of direct indirect observations, smoothing, and related topics.

B. Practical theory of equations treats iteration and the regula falsi, Newton's, Horner's and Graeffe's method as well as other means of solving equations in one unknown. Also Gauss's, Banachiewicz' and Cholesky's methods of elimination are given for linear systems. In addition the methods of iteration and relaxation are applied to linear and non-linear systems.

C. The calculus of finite differences presents divided differences, finite differences, interpolation, numerical integration, singular integrals, and trigonometric interpolation.

D. Initial value problems deal with those problems involving ordinary and partial differential equations where the desired solution is defined by the initial values. Graphical and numerical methods are treated, as is the Runge-Kutta method, interpolation method, extrapolation method, and others. The propagation of errors, questions of stability, etc., receive attention.

E. Boundary value and characteristic value problems, are treated by series, by the method of perturbations, by iteration, and by relaxation. The Ritz method and the Trefftz method likewise are shown.

F. Integral- and functional equations are handled by various methods such as summation, iteration, etc. The cases of degenerate kernels and singular equations are also treated. *W. E. Milne* (Corvallis, Ore.).

**Rall, L. B. Error bounds for iterative solutions of Fredholm integral equations.** Pacific J. Math. **5** (1955), 977–986.

Iterative methods for solving the integral equation $y(s)=x(s)-\int_a^b K(s,t)x(t)dt$ are studied from the general point of view of a complete normed linear space $X$ with elements $x, y, \cdots$, where the equation $Fx=y$ with $F$ as a linear operator in $X$ is to be solved. With respect to any linear operator $T$ and to the bounds $M(T)=\sup(\|Tx\|/\|x\|)$; $m(T)=\inf(\|Tx\|/\|x\|)$ two theorems are presented: 1) A unique solution of $Fx=y$ exists if and only if a linear operator $P$ together with its inverse $P^{-1}$ exists such that $M(I-PF)<1$ with $I$ denoting the unit operator; the solution can be written as $x=\sum_{n=0}^{\infty}(I-PF)^nPy$. 2) If a unique solutions exists and if there is a linear operator $P$ admitting $M(I-PF)<1$, then the iterative process $x_n=(I-PF)x_{n+1}+Py$ converges to the solution $x$ for any initial $x_0$ (total convergence) and its error is bounded by $\|x-x_n\|\leq[M(I-PF)]^n\|x-x_0\|$ and

$$\|x-x_n\|\leq M(I-PF)\cdot m^{-1}(PF)\cdot\|x_n-x_{n-1}\|.$$

The author uses these theorems for a review of iterative procedures as proposed by G. Wiarda, H. Bückner, C. Wagner and P. A. Samuelson. He reformulates conditions of total convergence and gives estimates for the number of iterations in a special numerical example. He also mentions that error bounds for two of the above mentioned methods were found by M. Schönberg; Schönberg too formulated such bounds in the language of Banach-spaces. *H. F. Bückner* (Schenectady, N.Y.).

Have

★**Albert, G. E. A general theory of stochastic estimates of the Neumann series for the solutions of certain Fredholm integral equations and related series.** Symposium on Monte Carlo methods, University of Florida, 1954, pp. 37–46. John Wiley and Sons, Inc., New York; Chapman and Hall, Limited, London, 1956. $7.50.

In diesem Auszug aus einem längeren Bericht [Oak Ridge Nat. Lab. Rep. **ORNL-1508** (1953)] werden Schätzfunktionen für die Funktionswerte der Lösung der Gleichung

$$\varphi(x)=g(x)+\int K(x,y)\varphi(y)dy$$

mit $g\geq 0$, $K\geq 0$ und konvergenter Neumann'scher Reihe diskutiert, die aus einer Folge $X_1, X_2, \cdots, X_N$ zufälliger Variabler, deren Länge $N$ selbst eine zufällige Größe ist, entstehen. Die unverzerrten (unbiased) Schätzfunktionen werden dabei durch gewisse bedingte Wahrscheinlichkeitsdichten der $X_\nu$ festgelegt. Eine Spezialisierung wird im Fall, daß eine Näherungslösung bekannt ist, empfohlen. *D. Morgenstern* (Berlin).

**Durand, David. A note on matrix inversion by the square root method.** J. Amer. Statist. Assoc. **51** (1956), 288–292.

The authors points out cases in matrix inversion by the square root method in which the inverse of the original symmetric matrix $A$ is not required so that one can eliminate the formal calculation and recording of $A^{-1}$, proceeding directly from the triangular matrix $S$ or its inverse given by the square root method which reduces $A$ to $SS'$. In discussing cases in which $S$ or $S^{-1}$ is of interest in its own right he points that if $A$ is a correlation matrix the off diagonal elements of $S$ are semi-partial correlation coefficients. *C. C. Craig* (Ann Arbor, Mich.).

**Bolton, H. C.; and Scoins, H. I. Eigenvalues of differential equations by finite-difference methods.** Proc. Cambridge Philos. Soc. **52** (1956), 215–229.

The paper is concerned with the Sturm-Liouville eigenvalue-problem $y''(x)+(\lambda-V(x))y(x)=0$; $y(0)=y(1)=0$, $0\leq x\leq 1$. The authors consider the difference method as defined by $u_{i+1}-2u_i+u_{i-1}+h^2\cdot(\Lambda-V_i)u_i=0$, where $h=1/n$, $V_i=V(ih)$ and $u_0=u_n=0$. It is assumed that $V(x)$ is analytic and that the difference-equation problem $u(x+h)-2u(x)+u(x-h)+h^2(\Lambda-V(x))u(x)=0$, $u(0)=u(1)=0$, admits convergent power series $\Lambda=\sum_{s=0}^{\infty}\nu_s h^s$, $u=\sum_{s=0}^{\infty}\phi_s(x)\cdot h^s$, for its eigenelements. From these assumptions it is derived that $\Lambda=\lambda+\nu_2h^2+O(h^4)$. This law is used in the sense of Richardson's deferred approach to the limit (extrapolation to $h=0$) and by means of a Neville table to calculate approximate eigenvalues of high accuracy from the determinant of the system for the values $u_i$. A formula for $\nu_2$ is given. The calculation of eigenfunctions is also dealt with; the substitution of $y''$ by a difference operator of order higher than two is shortly discussed. Numerical examples refer to Schrödinger's equation of one space dimension and deal with the particle in a spherical box, with the compressed and with the bounded hydrogen atom. The authors refer to a paper by Hans Bückner [Math. Z. **51** (1948), 423–465; MR **11**, 58]. They state that this paper shows the law $\Lambda=\lambda+O(h^2)$ and they believe that their proof of the law is simpler; they also claim that they go to higher accuracy beyond this law. {The facts are, that the latter paper does not assume the existence of the coefficients $\nu_s$ but proves the existence of some of them under assumptions on $V(x)$. It is shown that $\nu_0$, $\nu_1$, $\nu_2$ exist if $V(x)$ admits continuous derivatives up to second order; it is shown that $\nu_3$ and $\nu_4$ exist if the derivatives exist continuously up to order four. It is stated that $\nu_1=\nu_3=0$ while formulas for $\nu_2$ and $\nu_4$ are given (see § 12 of the paper). A special difference method with $\nu_2=0$ is discussed in § 14. The extrapolation to $h=0$ is dealt with in § 13 and applied in the numerical examples of § 16.} *H. F. Bückner.*

**Fettis, Henry E. Concerning the eigenvalues of a differential equation in convective heat transfer.** Quart. Appl. Math. **14** (1956), 112–114.

The system for which eigenvalues are wanted is $d^2f/dy^2+\lambda y(\frac{1}{2}-y)f=0$ with $f(0)=f(\frac{1}{2})=0$. The author transforms this system to $f''+\alpha^2(1-t^2)f=0$ with $f(\pm 1)=0$, expands the solution in powers of $\alpha$ with coefficients which are polynomials in $t$, and from this obtains the lowest eigenvalue to six figures, the second and third to about three and two places respectively. *W. E. Milne.*

**Fettis, Henry E. Concerning the eigenvalues of a differential equation in convective heat transfer.** Quart. Appl. Math. **14** (1956), 198–200.

L'auteur applique au problème

$$f''+\lambda y(\tfrac{1}{2}-y)f=0,\ f(0)=f(\tfrac{1}{2})=0,$$

la méthode de détermination des valeurs propres qui consiste à développer la solution satisfaisant à $f(0)=0$ suivant les puissances du paramètre. Il donne la première valeur propre avec 6 chiffres. Aucune étude théorique n'est faite. *J. Kuntzmann* (Grenoble).

**Grebenyuk, D. G. Construction of formulas of approximate computation of triple integrals in a region (D) representing a sphere.** Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **13** (1954), 43–55. (Russian)

**Grebenyuk, D. G. Construction of formulas of approximate computation of triple integrals on a region representing an ellipsoid.** Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **13** (1954), 57–69. (Russian)

The author obtains formulas of the type

$$\text{(1)}\quad \int_{(D)} f(x,y,z)dxdydz=\sum_{k=1}^{\mu} A_k f(x_k,y_k,z_k)+R_\mu,$$

in which $D$ is a region bounded either by a sphere or by an ellipsoid and $(x_k, y_k, z_k)$, are properly chosen points within the region. The selection of these points is based on an earlier paper by the author in same Trudy **6** (1951). *W. E. Milne* (Corvallis, Ore.).

**Borovskiĭ, P. V. On the exactness of mechanical-quadrature formulas in problems of determining displacements.** Kiev. Avtomobil.-Dorož. Inst. Trudy **2** (1955), 170–175. (Russian)

This is an elementary discussion of the errors committed in applying the 3-, 4-, and 5-, point Simpson-Cotes formulas to the integral $\int M_i M_k/EI\, dS$. *W. E. Milne.*

**Adachi, Ryuzo. Approximate formulas for definite integrals and differential coefficients.** Kumamoto J. Sci. Ser. A. **2** (1955), 196–209.

The integrals $F_i=\int_a^{x_i} f(x)dx$ $(x_i=a+ih;\ i=0, 1, 2, \cdots, p)$ are calculated by means of formulas $F_i\approx\sum_{j=0}^{p} A_{ij}f(x_j)$. In the same way the author calculates the derivatives by means of the formulas $f'(x_i)=\sum_{j=0}^{p} B_{ij}f(x_j)$ and in a similar way derivatives of higher order. The coefficients $A_{ij}$, $B_{ij}$ are such that the formulas become exact for polynomials $f(x)$ up to order $p$. Tables of coefficients up to $p=6$ for the integrals, up to $p=9$ for first order derivatives and up to $p=8$ for second order derivatives are presented. Formulas for a general $p$ are also dealt with. {The reviewer believes that the use of the theory of interpolation polynomials would have facilitated the calculation of the coefficients.} *H. F. Bückner.*

**Servais, F. Sur l'estimation des erreurs dans l'intégration numérique des équations différentielles linéaires du second ordre.** Ann. Soc. Sci. Bruxelles. Sér. I. **70** (1956), 5–8.

The author studies the propogation of errors in the numerical solution of $y''=yf(x)+g(x)$ by a method of approximate integration using symmetric differences. It is shown that for sufficiently small intervals a truncation error of the order of $h^4$ is superimposed on the inherited error. For $f(x)>0$ the latter increase exponentially with the number of the intervals and has a coefficient of the order of $h^5$. For $f(x)<0$ the inherited error is oscillating with an amplitude of the order of $h^4$ *W. E. Milne.*

**Mikeladze, Š. E. Numerical integration of differential equations in the complex plane.** Soobšč. Akad. Nauk Gruzin. SSR **17** (1956), 97–102. (Russian)

The author uses the same stype of quadrature formulas, applicable in the real case, to integrate numerically an ordinary differential equation of second order along some straight line in the complex plane. He also indicates the use of such methods for analytic continuation. *W. E. Milne* (Corvallis, Ore.).

**Giger, Adolf. Ein Grenzproblem einer technisch wichtigen nichtlinearen Differentialgleichung.** Z. Angew. Math. Phys. **7** (1956), 121–129.

The author gives a numerical procedure for computing the critical value $\alpha(\beta)$ of the damping coefficient $\alpha$ in the equation

$$(*)\qquad \frac{d^2\theta}{dt^2}+\alpha\frac{d\theta}{dt}+\sin\theta=\beta$$

above which the equation (*) has no periodic solution of the second kind. The procedure involves no step-wise numerical integrations and appears to be much simpler than that given by Urabe [J. Sci. Hiroshima Univ. Ser. A. **18** (1955), 379–389; MR **17**, 618]. The author's results, which are presented graphically, agree with those of Urabe. *C. E. Langenhop* (Ames, Iowa).

**Vorob'ëv, L. M. Applicability of S. A. Čaplygin's method of approximate integration to a certain class of ordinary nonlinear differential equations of second order.** Uspehi Mat. Nauk (N.S.) **11** (1956), no. 1(67), 181–185. (Russian)

The method of Čaplygin, when applicable, depends on the possibility of finding upper and lower dominating functions for the desired solution. These are solutions of auxiliary equations obtained from the given equation. In the present paper the author determines conditions under which Čaplygin's method is applicable to the second order equation of the form $dy'/dx=F(x, y, y')$. *W. E. Milne* (Corvallis, Ore.).

★**Bergman, Stefan. Some methods for solutions of boundary-value problems of linear partial differential equations.** Proceedings of Symposia in Applied Mathematics. Vol. VI. Numerical analysis, pp. 11–29. Published by McGraw-Hill Book Company, Inc., New York, 1956 for the American Mathematical Society, Providence, R. I. $9.75.

The Bergman kernel function and Bergman's integral operators possess considerable potential value in numerical analysis, but this potential has remained largely unexplored as yet. The present expository paper aims to call attention to this situation, and serves as an intro-

duction to the author's extensive bibliography of original contributions in this area. Topics mentioned include: construction of the kernel function

$$k(X, T)=\sum_{n=0}^{\infty}\psi_n(X)\psi_n(T),\ X\in B,\ T\in B,$$

for a second order elliptic partial differential equation $L(\psi)=0$ on a domain $B$ (where the $\psi_n$ are a complete orthonormal set of particular solutions of $L(\psi)=0$ over $B$), solution of boundary value problems using this kernel function, examples of some typical complete sets of solutions of the type needed to construct $k(X, T)$, and determination of the constants in the Schwarz-Christoffel transformation by means of a kernel function constructed from a complete orthonormal set of analytic functions in the class $L^2(B)$ (where now $B$ denotes the polygonal domain being mapped). Working in the pseudo-logarithmic plane, the author obtains explicit solutions for subsonic compressible flows in the case where the domain in the physical plane is bounded by straight line segments and free boundaries. The Bers-Gelbart-Bergman functions $\phi_{n,1}$, $\psi_{n,1}$, $\phi_{n,2}$, and $\psi_{n,2}$ are used to solve the initial value problem

$$\phi_\theta=\psi_H,$$

$$\phi_H=-l(H)\psi_\theta,\ \psi(0,\theta)=X_1(\theta),\ \frac{\partial\psi(H,\theta)}{\partial H}\bigg|_{H=0}=X_2(\theta),$$

where $l$, $X_1$, and $X_2$ are known (this corresponds to an initial value problem on the sonic line in compressible flow). Bergman's integral operators are then used to solve this same initial value problem by a second method which has the additional virtue of dealing also with the case where $\psi(H,\theta)$ is multivalued. Finally, the extension of integral operators to the case of partial differential equations in three independent variables is presented.

*R. B. Davis* (Syracuse, N.Y.).

**Bauhuber, Franz. Fehlerabschätzungen und Verbesserungen der numerischen Charakteristikenmethoden für Anfangswertprobleme in der Gasdynamik.** Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1955**, 23-43 (1956).

In this note are collected the results of the author's Munich dissertation of the same title (1954). The characteristics nets corresponding to plane and axisymmetric steady supersonic flow are considered. In the axissymetric case points at the axis are not considered. Isentropic flow is first assumed, and methods of correction "to third order" are given. For nonisentropic flow only a first-order approximation is used; i.e., one that makes use of characteristic quantities at the preceding mesh points only, without further interpolation or averaging. In the dissertation, it is stated, a second approximation is given. Next, results are stated which give the error of mesh-point coordinates in terms of the differences between these coordinates as calculated with two different mesh sizes. Finally some numerical results obtained in two typical cases are presented. {The reviewer notes that another study of the accuracy of the method of characteristics for plane flow has recently appeared, viz., M. G. Hall, Quart. J. Mech. Appl. Math. **9** (1956), 320–333.}

*W. R. Sears* (Ithaca, N.Y.).

**Stallman, Robert W. Numerical analysis of regional water levels to define aquifer hydrology.** Trans. Amer. Geophys. Union **37** (1956), 451–460.

**See also:** Parodi, p. 4; Bruck, p. 20; de Greiff Bravo, p. 32; Krylov, p. 32; Baženov, p. 34; Moiseev, p. 41; Collatz, p. 46; Douglas, p. 46; Kislicyn, p. 48; Lax and Richtmyer, p. 48; Gubler, p. 76; Ansermet, p. 65; Mertens, p. 102; Wunderlich, p. 80; Chow, p. 87; Barenblatt, p. 91; Seide, p. 84; Ham and Ruedenberg, p. 98; Jaeger, p. 94.

### *Machines and Modelling*

★**Lehmer, Emma. Number theory on the SWAC.** Proceedings of Symposia in Applied Mathematics. Vol. VI. Numerical analysis, pp. 103–108. Published by McGraw-Hill Book Company, Inc., New York, 1956 for the American Mathematical Society, Providence, R. I. $9.75.

The author's account includes three classes of problems tried on the SWAC: (1) Special computations (as the testing of Fermat primes, Mersenne primes, or difference sets); (2) Testing of hypotheses (such as Riemann's, Polya's, Turan's, Fermat's) unfortunately with no counterexamples occurring; (3) Research problems (computational explorations into Jacobsthal's sums or Ramanujan's function) often leading to theorems of Lehmer.

*Harvey Cohn* (St. Louis, Mo.).

**Hartley, H. O. A plan for programming analysis of variance for general purpose computers.** Biometrics **12** (1956), 110–122.

The organization of computations for analyzing a $k$-factor experiment is described in detail for $k=3$, and it is shown how the analysis of other designs can be reduced to the factorial. As the author admits, in some cases the reduction might be more complicated than the calculations programed, which are basically only sums and sums of squares. *A. S. Householder* (Oak Ridge, Tenn.).

**Riguet, Jacques. Algorithmes de Markov et théorie des machines.** C. R. Acad. Sci. Paris **242** (1956), 435–437.

The concept of "normal algorithm" in the sense of A. A. Markov [Trudy Mat. Inst. Steklov. **42** (1954); MR **17**, 1038] is related to the author's general theory [C. R. Acad. Sci. Paris **237** (1953), 425–427; MR **15**, 559] of Ashby machines by an abstract "flow diagram." *G. Birkhoff.*

**Barnett, M. P.; Robertson, H. H.; and Albasiny, E. L. High-speed computation of molecular integrals.** J. Chem. Phys. **25** (1956), 367–368.

# PROBABILITY

### *Theory of Probability*

Have

★**Laning, J. Halcombe, Jr.; and Battin, Richard H. Random processes in automatic control.** McGraw-Hill Book Company, Inc., New York-Toronto-London, 1956. ix+434 pp. $10.00.

The book is the outgrowth of lectures delivered by the authors at the M.I.T. The reader not being assumed to have a probabilistic background, a brief introduction (which contains, however, an illuminating discussion of the use and abuse of the mean-squared-error criterion) is followed by a long Chapter 2 on the fundamental concepts of probability theory. This chapter is soundly based on the theory of measure and is, generally speaking, more

thorough than one would expect in such a book. Beside excellent examples, which are a feature of the whole treatise, Chapter 2 contains, for instance, an explicit expression for the general mixed moments of a normal multivariate distribution. It is a pity, though, that discrete distributions are described by means of the henceforth ubiquitous "delta function". Chapter 3 brings a fairly elementary "statistical description of random processes". Surprisingly enough, the authors confine themselves to the case of a continuous parameter; when essentially discrete processes appear, as in Exercise 3.2-1, continuity is introduced by means of artificial devices. The definition of ergodicity is loosely worded, and subsequently no attempt is ever made to prove that any particular process is ergodic. Very little attention is paid to problems of estimation, called "empirical determination". The reader is not warned against applying ordinary harmonic analysis to stochastic processes. On the contrary, the authors' definition of the spectral density function of a stochastic process presupposes the existence of a limit of the continuous counterpart of the periodogram for sample sizes tending to infinity, while, at least in the case of a discrete parameter and according to Grenander [Ark. Mat. **1** (1952), 503–531; MR **14**, 187], this limit does not exist, even in the shape of a random variable.

Chapter 4 is devoted to processes based on Gaussian and Poisson distributions; white noise is described, but not rigorously defined. Chapter 5 is concerned with the effect of linear filters with constant coefficients on stationary processes, and Chapter 6 with the analysis (largely confined to second moments) of non-stationary processes. Chapter 7 deals with smoothing and prediction, and offers a simplified version of the theory evolved by N. Wiener [Extrapolation, interpolation, and smoothing of stationary time series, Wiley, New York, 1949; MR **11**, 118], as well as an account of the approach devised by H. W. Bode and C. E. Shannon [Proc. I.R.E. **38** (1950), 417–425; MR **11**, 672]. The case, important in applications, of rational spectral density functions, is treated in detail. The eighth, and last, chapter deals with the case when the realization of a process is known only over a finite time interval; it is an attempt at a unified treatment of various special cases in which the problem has been solved. The appendices deal mainly with computation techniques. The bibliography is rudimentary on the probabilistic side, but full and up-to-date as far as engineering applications are concerned.

Apart from some inadmissible neologisms, such as "randomnicity" for randomness and "optimalization" for optimization, the book reads well. It contains a great wealth of material, often not readily accessible elsewhere, and in several instances contributed by the authors themselves.

*S. K. Zaremba* (Wolverhampton).

**Castoldi, Luigi. Un teorema fondamentale nella teoria probabilistica degli eventi ricorrenti.** Atti Accad. Ligure **11** (1954), 185–191 (1955).

The author discusses recurrent events, making what he considers to be a generalization of Feller's definition [An introduction to probability theory and its applications, v. 1, Wiley, New York, 1950; MR **12**, 424]. {Since it is easily seen that his definition is equivalent to that of Feller, his results are not new.{

*J. L. Doob* (Geneva).

**Oshio, Shigeru. On mean values and geometrical probabilities in $E_n$.** Sci. Rep. Kanazawa Univ. **3** (1955), 199–207.

Generalization to the $n$-dimensional euclidean space of the results previously given for $n=3$ [same Rep. **3** (1955), no. 1, 35–43; MR **17**, 274]. (Note: in the last review the index $r$ must be replaced by $p$.)

*L. A. Santaló.*

**Dvoretzky, Aryeh. On covering a circle by randomly placed arcs.** Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 199–203.

The relationship between the probability of covering a unit circle by random arcs and the sum of the lengths of these arcs is studied.

*S. C. Moy* (Detroit, Mich.).

**Fréchet, Maurice. On some consequences of information concerning a priori probabilities.** J. Osaka Inst. Sci. Tech. **4** (1952), 25–38. (Esperanto)

In the binomial case, let upper and lower bounds for the cumulative prior probability distribution function be given. Denote the cumulative posterior probability function by $\omega(z)$ $(0\leqq z\leqq 1)$. For a particular value $z_1$, the author investigates whether an admissible cumulative prior probability function exists for which $\omega(z_1)$ attains a maximum. If so, he gives a formula for such a function. For some values $z_1$ such a function does not exist, failing by not being everywhere continuous on the right as desired (though the function is continuous on the left). In these cases the author gives a sequence of prior probability functions for which the limit of $\omega(z_1)$ is the least upper bound.

*A. Blake* (Royal Oak, Mich.).

**Kawata, Tatsuo. A renewal theorem.** J. Math. Soc. Japan **8** (1956), 118–126.

Let $\{X_n\}$ be independent random variables with $0<EX_n<\infty$ and cdf's $F_n$ satisfying $\int_{-\infty}^0 e^{-sx}dF_n(x)<\infty$ for $0\leqq s\leqq s_0$ for some $s_0>0$, $\int_A^\infty x\,dF_n\to 0$ as $A\to\infty$ uniformly in $n$ and $\int_{\infty}^{-A} e^{-sx}dF_n(x)\to 0$ as $A\to\infty$ uniformly in $n$ and $s$, $0\leqq s\leqq s_0$. If $E(X_1+\cdots+X_n)/n\to m>0$ as $n\to\infty$, then

$$x^{-1}\int_0^x \Big(\sum_n \Pr\{x<X_1+\cdots+X_n\leqq x+h\}\Big)dx\to\frac{h}{m}\text{ as } x\to\infty.$$

*D. Blackwell* (Berkeley, Calif.).

**Takahashi, Shigeru. On the series of some independent random variables.** Sci. Rep. Kanazawa Univ. **3** (1955), 209–212.

Let $\{X_n\}$ be a sequence of independent random variables with a common distribution function $F$, and let $\{a_n\}$ be a decreasing sequence of positive numbers. The following are typical of the author's results. Suppose that $\sum P\{a_n|X_n|>1\}<\infty$. Then if, for some $\lambda>1$,

$$(*)\qquad \frac{a_{n+1}}{a_n}\leqq\Big(\frac{n+1}{n}\Big)^\lambda,$$

it follows that the series $\sum a_nX_n$ converges absolutely, with probability 1. If $F$ is symmetric, and if (*) holds for some $\lambda>\frac{1}{2}$, then the series converges with probability 1. The limiting exponents are the best possible ones.

*J. L. Doob* (Geneva).

**Schmid, Paul. Sur les théorèmes asymptotiques de Kolmogoroff et Smirnov pour des fonctions de distribution discontinues.** C. R. Acad. Sci. Paris **243** (1956), 349–351.

Let $\{X_n\}$ be a sequence of mutually independent ran-

dom variables with a common distribution function $F$, and let $S_N$ be the empirical distribution function of $X_1, \cdots, X_N$. The author generalizes the Kolmogorov-Smirnov theorems, which hypothesize the continuity of $F$, by giving expressions for the limiting distributions of $N^{\frac{1}{2}} \sup_x |S_N(x)-F(x)|$ and $N^{\frac{1}{2}} \sup [S_N(x)-F(x)]$ which are valid in all cases. Proofs are omitted. *J. L. Doob.*

**Yntema, L. An elementary proof of the central limit theorem.** Verzekerings-Arch. Actuarieel Bijvoegsel 33 (1956), 19*–40*.

The author gives a complete and simple proof of the Central Limit Theorem based only on knowledge of the Riemann-Stieltjes integral. All preliminary ideas are presented so that this paper would be of interest to people who have had no mathematics beyond calculus.
*J. Blackman* (Ithaca, N.Y.).

**Blanc-Lapierre, André; Dumontet, Pierre; et Savelli, Michel. Remarques sur quelques propriétés de fonctions aléatoires stationnaires intervenant dans des problèmes de changement de fréquence.** C. R. Acad. Sci. Paris 242 (1956), 2799–2800.

The note refers to the concept of total stationarity ("stationnarité totale d'ordre $K$" [cf. Blanc-Lapierre and Fortet, Théorie des fonctions aléatoires, Masson, Paris, 1953; MR 15, 883]). Sufficient conditions are stated for the total stationarity of order $K$ of a stochastic process $X(t)$ implying the same property in the "frequency-shifted" process $X(t)e^{2\pi i\nu_0 t}$, or in some components of the spectral representation of $X(t) \cos 2\pi\nu_0 t$. *S. K. Zaremba.*

**Pál, L. On the theory of stochastic processes in cosmic radiation.** Ž. Eksper. Teoret. Fiz. 30 (1956), 362–366; supplement to 30, no. 2, 7. (Russian. English summary)

Let $\xi(t)$ be a stochastic process depending on a continuous time parameter and assume that $\xi(t)$ has jumps at certain randomly distributed instants and that $\xi(t)$ varies between consecutive jumps in accordance with some deterministic law. This law is described by means of a function $f(x, u)$ which has the following meaning: If $\xi(t)=x$ then $\xi(t+u)=f(x, u)$ on condition that no jumps occurred in the time interval $[t, t+u]$. If the process is a Markov process then the function $f(x, u)$ must satisfy a functional equation. The Fokker-Plank equations are derived for such a process. This is extended to the case where $\xi(t)$ is a random vector. *E. Lukacs.*

**Yanoši [Janossy], L. Generalized form of the diffusion equation for a single particle.** Ž. Eksper. Teoret. Fiz. 30 (1956), 351–361; supplement to 30, no. 2, 7. (Russian. English summary)

The author studies a diffusion process where the variables describing the state are not necessarily independent. He admits instantaneous changes of state caused by collisions as well as changes taking place between collisions and derives the Fokker-Planck equations for such a process. *E. Lukacs* (Washington, D.C.).

**Fuchs, Aimé; et Vigier, Jean-Pierre. Tendance vers un état d'équilibre stable de phénomènes soumis à une évolution markovienne.** C. R. Acad. Sci. Paris 242 (1956), 1120–1122.

The following theorem is announced and the proof is sketched. Let $X(t)$ be a Markoff process with transition function
$$F(t, x; \tau, E)=P\{X(\tau) \in E | X(t)=x\} \quad (t\leqq\tau)$$
and probability distribution at instant $t$
$$\phi(t, E)=P\{X(t) \in E\} \quad (t\geqq 0).$$
If 1) there exists a probability distribution $\Lambda(E)$ such that
$$\Lambda(E)=\int_{-\infty}^{+\infty} \Lambda(dx)F(t, x; \tau, E)$$
for all $t$, $\tau\geqq 0$, $t\leqq\tau$ and $\Lambda(E)>0$ whenever $\text{mes}_B(E)>0$, 2) $\phi(0, E)\leqq K\Lambda(E)$ $(0<K<\infty)$, 3) there exists a Borel set $C$ with $\Lambda(C)>0$ such that for sufficiently large $t$, one has
$$F(t, x; \tau, E)\geqq\delta\Lambda(E) \quad (\delta>0)$$
for every $t \in E$ and every $x \in C$, then $\phi(t, E)$ converges to $\Lambda(E)$ as $t\to\infty$. The convergence is uniformly in $E$ and exponentially fast. *S. C. Moy* (Detroit, Mich.).

**Gubler, Hermann. Über eine allgemeine Methode der Lösung des Zinsfussproblems für verschiedene Versicherungsformen und die Darstellung der darin auftretenden Momente.** Mitt. Verein. Schweiz. Versich.-Math. 56 (1956), 91–144.

The method referred to in the title was proposed by E. Zwinggi [Skand. Aktuarietidskr. 33 (1950), 88–97; Bl. Deutsch. Ges. Versicherungsmath. 1 (1952), no. 3, 105–113; MR 12, 135; 15, 68] who illustrated it by the example of the net premium for an endowment insurance. The author applies this method to both a variable gross premium and the net value of an endowment insurance. To avoid the separate treatment of the discrete and continuous case he applies, whenever he can do it, the concept of integration introduced by the reviewer for the unification of these cases [Mitt. Verein. Schweiz. Versich.-Math. 43 (1943), 127–179; Portugal. Math. 4 (1944), 73–118; MR 6, 120]. *H. M. Schaerf* (St. Louis, Mo.).

**Doob, J. L. A probability approach to the heat equation.** Trans. Amer. Math. Soc. 80 (1955), 216–280.

The Dirichlet problem for the heat equation on an arbitrary open domain is solved, by setting, by means of the parabolic measure defined probabilistically, the value of the function at a point of its domain equal to the average value of the boundary function.

After preparing the sub- and superparabolicity of a function in $N+1$ space of points $z=(\xi_1, \xi_2, \cdots, \xi_N; s)$, the the notion of the parabolic measure is introduced: Let $z=(\xi; s)$ be a point of a finite open set in $N+1$ space and let $S$ be the boundary of $D$. Denote by $Z(z, S)$ the point $\in S$ where the Brownian trajectory $z(t)=(\xi+x(t); s-t)$, defined by the $N$-dimensional Brownian motion $x(t)$ $(0\leqq t<\infty)$ first meets $S$. The distribution of $Z(z, S)$ defines a parabolic measure on the Borel subsets of $S$. In terms of such parabolic measure, the Dirichlet problem for the heat equation is discussed in detail. Thus the finite boundary point $z_0$ is regular if and only if
$$\lim_{z\uparrow z_0,\, z\in D} Z(z, S)=z_0$$
in probability. Here $z=(\xi; s) \uparrow z_0=(\xi_0; s_0)$ means that $s<s_0$ and $z\to z_0$. It is proved that almost no Brownian trajectory from a point of a domain first meets the boundary in an irregular point, and, in turn, an irregular point is characterized by the fact that almost all Brownian trajectories from the point enter the domain at once and stay there for some time interval. In this way, it is shown that the most natural curves of approach to the boundary along which sub- or superparabolic functions have limits are certain probability trajectories which do not depend on the domain in question. *K. Yosida.*

**Hunt, G. A. Some theorems concerning Brownian motion.** Trans. Amer. Math. Soc. 81 (1956), 294–319.

Given a Markov process with stationary transition probabilities, the author is interested in certain random times such that, when the associated state is known, the future is a new Markov process which is governed by the original transition probabilities and is independent of the past. We shall be content to report a somewhat special version of his statements.

Consider a Brownian motion with state space $R^n$ and continuous sample functions $X=(X(t):t\geqq 0)$, built over the probability field $(\Omega, \mathcal{F}, \mathcal{P})$, let $T$ be a $(\mathcal{P})$ measurable random time and let $\Omega'=(T<+\infty)$ have positive $\mathcal{P}$ measure, let $\mathcal{F}'=(B: B=A\cap\Omega',\ A\in\mathcal{F})$ and $\mathcal{P}'(\cdot)=\mathcal{P}(\cdot/\Omega')$, and call $T$ a Markov time if, over the new probability field $(\Omega', \mathcal{F}', \mathcal{P}')$, $Z=(X(t+T)-X(T): t\geqq 0)$ is a Brownian motion, independent of the stopped process $X^*=(X(\min(t, T)): t\geqq 0)$. We have the theorem: $T$ is a Markov time provided either that it be Borel measurable in $X^*$ or that it be the limit of Markov times $S$ such that $(S=+\infty)=(T=+\infty)$.

Let $E$ be closed in $R^n$ and let $E'$ be the points of $E$ which are regular for the associated Dirichlet problem. The author remarks that $T=\inf(t: X(t)\in E,\ t>0)\cup(t=+\infty)$ is a Markov time, and uses this to show that $q(t, r, s)=\mathcal{P}(T>t,\ X(t)\in ds/X(0)=r)/ds$ is the elementary solution for the heat equation

(1) $$v_t=(\tfrac{1}{2})\Delta v \text{ on } R^n-E,\ v=0 \text{ on } E',$$

and that $G(r, s)=\int_0^{+\infty} q(t, r, s)dt$ is the Green function for the Poisson equation

(2) $$(\tfrac{1}{2})\Delta v=-u \text{ on } R^n-E,\ v=0 \text{ on } E'.$$

When $n=2$ and $E$ has positive logarithmic capacity, he uses these expressions to estimate the speed at which the solution $v(t, s)$ of

(3) $$v_t=(\tfrac{1}{2})\Delta v,\ v(0, \cdot)=0 \text{ on } R^n-E,$$

$v(t, \cdot)$ prescribed on $E'$, converges to the corresponding solution $v(s)$ of $\Delta v=0$. The result is

$$[v(s)-v(t, s)]\log t\to 2\pi G(s, \infty)v(\infty), \text{ as } t\uparrow +\infty,$$

and it follows that

$$\mathcal{P}(T>t/X(0)=s)\sim 2\pi G(s, \infty)/\log t, \text{ as } t\uparrow +\infty,$$

which had been conjectured by M. Kac.

The reader will find it profitable to compare the present paper with Doob's recent work on the heat equation [see the paper reviewed above]. *H. P. McKean, Jr.*

**Urbanik, K. On a problem concerning birth and death processes.** Acta Math. Acad. Sci. Hungar. 7 (1956), 99–106. (Russian. English summary)

Denote by $S(\xi)$ the maximal number of particles in the homogeneous birth and death process $\xi_t$. In this paper we consider the expectation of the following random variable:

$$\varrho(\xi)=\sup_{\xi_t=S(\xi)} t-\inf_{\xi_t=S(\xi)} t.$$

The conditional expectation of the random variable $\varrho(\xi)$ under the assumption $S(\xi)=n$ is given by the formula

$$E(\varrho(\xi)|S(\xi)=n)=\frac{1}{a_0}\sum_{k=1}^{n}\frac{U_{k-1}}{k}\left(\frac{U_{n-k}}{U_n}-\frac{U_{n-k-1}}{U_{n-1}}\right),$$

where for $n\geqq 1$

$$U_n=(-a_0)^{-n}\begin{vmatrix} a_1 & a_0 & 0 & 0 & \cdots & 0\\ a_2 & a_1 & a_0 & 0 & \cdots & 0\\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot\\ a_n & a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1\end{vmatrix},$$

$U_0=1$, $U_n=0$ for $n<0$ and $a_0, a_1, \cdots, a_n, \cdots$ denote the transition intensities of the process $\xi_t$.

Let $P$ denote the probability of extinction of the process $\xi_t$. If $\sum_{n=0}^{\infty} n^3a_nP^n<\infty$, then the asymptotic formula

$$E(\varrho(\xi)|S(\xi)=n)=\begin{cases} -\dfrac{1}{nm_1}+o\left(\dfrac{1}{n}\right) & \text{for } m_1\neq 0,\\ \dfrac{1}{m_2}+o(1) & \text{for } m_1=0\end{cases}$$

is true where

$$m_1=\sum_{n=0}^{\infty} na_nP^{n-1} \text{ and } m_2=\sum_{n=0}^{\infty} n(n-1)a_nP^{n-2}.$$

(Author's summary.) *J. Wolfowitz* (Ithaca, N.Y.).

**Gander, R. S. Operational research on queueing problems.** Research 9 (1956), 295–301.

A number of results in queueing theory, such as the distribution of waiting time for random service, or for bulk service, or for service with priorities, are reviewed for their interest in operational research. *J. Riordan.*

**Castoldi, Luigi. Sulla distribuzione dei tempi di estinzione nelle discendenze biologiche.** Boll. Un. Mat. Ital. (3) 11 (1956), 158–167.

Let

$$G(z)=G^{(1)}(z)=p_0^{(1)}+p_1^{(1)}z+p_2^{(1)}z^2+\cdots,$$

where $p_\nu^{(1)}$ is the probability that an individual will have $\nu$ offspring, only males being considered. Then if

$$G^{(k)}=G[G^{(k-1)}(z)]=p_0^{(k)}+p_1^{(k)}z+p_2^{(k)}z^2+\cdots,$$

$p_\nu^{(k)}$ is the probability that the same individual will have exactly $\nu$ descendents in the $k$th generation. The probability of extinction of the line at a specified generation and other parameters of the distributions can be computed from algorithms based upon this principle. {This use of iterated functions has been exploited by A. J. Lotka [Théorie analytique des associations biologiques, 2ième partie, Hermann, Paris, 1939], and, much more extensively, by C. J. Everett and S. Ulam in a declassified but, unfortunately, otherwise unpublished series of reports from Los Alamos.} *A. S. Householder.*

**See also:** Kubilyus, p. 17; Erdös, p. 18; Prékopa, Rényi and Urbanik, p. 25; Daniels, p. 54; Hunt, p. 54; Albert, p. 72; Landé, p. 95.

## STATISTICS

**Ramachandran, K. V. On the simultaneous analysis of variance test.** Ann. Math. Statist. 27 (1956), 521–528.

Let $S_1, S_2, \cdots, S_k$ and $S$ be mutually independent random variables having a $\chi^2$ distribution with $t_1, t_2, \cdots, t_k$ and $m$ d.f. respectively, and let

$$F_i=\frac{S_i}{S}\frac{m}{t_i}.$$

The author considers the distribution problem $P[F_i\leqq a_i,\ i=1, 2, \cdots, k]$; the multiple integral is reduced to a series expression which however would be tedious to use in tabulation. A simpler result is obtained for the special case $t_1=t_2=\cdots=t_k=t$ and $a_1=a_2=\cdots=a_k=a$. For this case a tabulation of the solutions of the equation

$$P[F_i\leqq a,\ i=1, 2, \cdots, k]=.95$$

is given for various values of $t$ and $m$.

Further if the $S_i$ have a non-central $\chi^2$ distribution with d.f. as above and with parameters of non-centrality $\lambda_i$ then $P[F_i \geq a_i, i=1, 2, \cdots, k]$ is a monotone function of each $\lambda_i$, i.e. the power function of the simultaneous analysis of variance test is monotone. *D. G. Chapman.*

**Hogg, Robert V. On the distribution of the likelihood ratio.** Ann. Math. Statist. 27 (1956), 529–532.

Let $P$ be a positive continuous function on $(a, \infty)$, let $Q(b)=1/\int_a^b P(x)dx$, and let $f_b$ be the density $Q(b)P$ for $a \leq x \leq b$, 0 elsewhere. If $x_1, \cdots, x_k$ are independent variables with densities $f_{b_1}, \cdots, f_{b_k}$ and $\lambda$ is the likelihood ratio for testing $b_1=\cdots=b_k=b_0$ specified, then $-2 \log \lambda$ has a $\chi^2$ distribution with $2k$ degrees of freedom under the null hypothesis. If $b_0$ is unspecified, $-2 \log \lambda$ has a $\chi^2$ distribution with $2(k-1)$ degrees of freedom. A similar result is obtained when $a$ is not fixed but is a decreasing function of $b$. *D. Blackwell* (Berkeley, Calif.).

**Hodges, J. L., Jr.; and Lehmann, E. L. The efficiency of some nonparametric competitors of the $t$-test.** Ann. Math. Statist. 27 (1956), 324–335.

Let $A$ and $A^*$ be two tests of the hypothesis $\theta=\theta_0$. If $A$ at level $\alpha$ using $N$ observations has power $\beta_A(N, \alpha, \theta)$ against alternative $\theta$, and $A^*$, also at level $\alpha$, requires $N^*$ observations to produce the same power at the same alternative, the efficiency of $A^*$ relative to $A$ in these circumstances is defined to be the ratio $N/N^* = e_{A*,A}(N, \alpha, \theta)$. Pitman has called $\lim_{N\to\infty, \theta\to\theta_0} e_{A*,A}(N, \alpha, \theta) = e_{A*,A}$ the asymptotic relative efficiency of $A^*$ with respect to $A$. In particular, given two populations with cdf's $F(u)$ and $F(u-\theta)$ respectively, let $A$ be the usual $t$-test of the hypothesis $\theta=0$ and $A^*$ be the Wilcoxon test of the same hypothesis. Pitman has shown (in unpublished lecture notes) that $e_{A*,A}=12\sigma^2[\int f^2(x)dx]^2$, where $f(x)=F'(x)$. The authors show that $e_{A*,A}$ never falls below $108/125=0.864$, so that from the point of view of Pitman's asymptotic efficiency the use of the Wilcoxon test instead of the customary $t$-test never entails a serious loss of efficiency when testing against shift. There is no lower bound for the relative efficiency of the Wilcoxon test in the case of contamination alternatives, i.e., when one population is distributed according to $F$, the other according to $(1-\theta)F+\theta G$, the null hypothesis again being $\theta=0$. The Wilcoxon test, like any rank test, is insensitive to the size of large deviations. As the authors point out, this is often rather an advantage than a disadvantage.

In addition to Pitman's concept of limiting efficiency, the authors discuss other possible limiting concepts as, e.g., $\theta\to\infty$ holding $N$ and $\alpha$ constant, or $N\to\infty$ holding $\alpha$ and $\theta$ constant. The latter limiting process is investigated in a comparison of the sign test with the one-sample $t$-test. The results are in good agreement with the simpler Pitman limit. *G. E. Noether.*

**Konijn, H. S. On the power of certain tests for independence in bivariate populations.** Ann. Math. Statist. 27 (1956), 300–323.

Pitman's method for determining the asymptotic relative efficiency of two tests is extended to the case of tests involving more than one parameter. The results are then used to study the asymptotic power and efficiency of several nonparametric correlation coefficients as well as of the ordinary (parametric) correlation coefficient in tests of independence of two variables against the alternative that the two variables are of the form $\theta_1 Y+\theta_2 Z$ and $\theta_3 Y+\theta_4 Z$, where $Y$ and $Z$ are independent. *G. E. Noether* (Boston, Mass.).

**Wishart, John. $\chi^2$ probabilities for large numbers of degrees of freedom.** Biometrika 43 (1956), 92–95.

The author derives asymptotic Cornish-Fisher type expansions for the $\chi^2$ distribution when $n$, the number of degrees of freedom, is large. In case of the direct expansion for the distribution of $\chi^2$, the preliminary transformation $w=(n/2)^{\frac{1}{2}} \ln (\chi^2/n)$ is used, while for the percentage points of $\chi^2$, $z=\frac{1}{2} \ln (\chi^2/n)$ is employed. These expansions should converge more rapidly than the well-known formulas of Campbell [Bell. System Tech. J. 2 (1923), no. 1, 95–113]. Actually the transformation

$$t=(\chi^2-n)/(2n)^{\frac{1}{2}}, \quad \alpha_3^2=8/n,$$

followed by a double interpolation in Salvosa's Type III table [Ann. Math. Statist. 1 (1930), no. 2, 191–198, 1–125] will provide the desired probabilities or percentage points for large $n$. *L. A. Aroian* (Culver City, Calif.).

**Bellman, Richard. A problem in the sequential design of experiments.** Sankhyā 16 (1956), 221–229.

There are two machines, one of which has a known probability $s$ of paying one unit when operated, the other having an unknown probability of $r$. If $r$ has initial a priori cdf $F$ on the unit interval and a payment of one unit on the $n$th operation has present value $a^n$, $0<a<1$, what policy for an infinite sequence of plays has the highest expected present value? If $f(m, n)$ denotes the highest expected present value when $F$ is replaced by the a posteriori distribution of $r$ after $m$ successes and $n$ failures on the unknown machine, a functional equation is given for $f(m, n)$. This equation is shown to have a unique solution bounded above by $1/(1-a)$, and an iterative method for approximating $f$ is given. The highest expected present value is $f(0, 0)$ and the method of achieving this is: operate the unknown machine whenever $f(m, n)>1/(1-a)$, otherwise operate the known machine. The smallest value $s(m, n)$ of $s$ for which $f(m, n)=1/(1-a)$ satisfies $s(m+1, n)>s(m, n)>s(m, n-1)$.

*D. Blackwell* (Berkeley, Calif.).

**Graybill, Franklin A.; and Wortham, A. W. A note on uniformly best unbiased estimators for variance components.** J. Amer. Statist. Assoc. 51 (1956), 266–268.

For variance component models in the analysis of variance (Model II) the authors state a theorem which is a direct consequence of the complete class theorem for sufficient estimators and which has the import that for all the balanced complete variance component models, the minimum variance unbiased estimators of linears functions of the expected mean squares are the same linear functions of the corresponding mean squares. The method of proof is indicated. *C. C. Craig* (Ann Arbor, Mich.).

**Leander, Erik K.; and Finney, David J. An extension of the use of the $\chi^2$-test.** Appl. Statist. 5 (1956), 132–136.

"A recent inquiry about the statistical analysis appropriate to a problem in paper manufacture led to a recognition of a simple extension of the use of $\chi^2$ in the analysis of sample statistics with Poisson distributions." (From the authors' summary.) *M. Dwass.*

**Wünsche, Günther. Ein Sequenz-Test zur Kontrolle von Ausscheidehäufigkeiten.** Mitt. Verein. Schweiz. Versich.-Math. 56 (1956), 77–89.

An expository account of the sequential testing of the

mean of a Poisson process (namely, $\mu_0 t$ vs. $\mu_1 t$, $t$ being time). The numerical application to a group of coeval lives makes the implicit assumption that each death is replaced by an identical live individual. *H. L. Seal.*

**Baker, G. A. The effects of wide groupings on the distributions of array means and variances for correlated normal variables.** Ann. Inst. Statist. Math., Tokyo **7** (1956), 103–106.

The author points out that rather wide groupings are often employed in processing normal bivariate data with the result that the grouped frequency arrays are no longer normal and the distributions of the averages and varainces of the grouped frequency arrays differ with the different groupings employed. *S. Kullback.*

**Castañs Camargo, Manuel; and Medina e Isabel, Mariano. The logarithmic correlation.** An. Real Soc. Españ. Fis. Quim. Ser. A. **52** (1956), 117–136. (Spanish. English summary)

Let $X$, $Y$ be random variables with values $1, 2, \cdots, m$; $1, 2, \cdots, n$ respectively, and let $p_i=\Pr(X=i)$, $q_j=\Pr(Y=j)$, $p_{ij}=\Pr(X=i, Y=j)$. Let $P=-\sum p_i \log p_i$, $Q=-\sum q_j \log q_j$, $S=-\sum p_{ij} \log p_{ij}$, denote the entropies of $X$, $Y$, $(X, Y)$ respectively. Then $\max(P, Q) \leqq S \leqq P+Q$, with equalities if and only if the variable with smaller entropy is a function of the other and $X$, $Y$ are independent, respectively {the authors do not use the term "entropy" and are apparently not familiar with Shannon's work, Bell System Tech. J. **27** (1948), 379–423, 623–656; MR **10**, 133.} Accordingly, the authors propose $r_p=(P+Q-S)/P$, $r_q=(P+Q-S)/Q$, $r=2(P+Q-S)/(P+Q)$ as replacements for the correlation coefficient. It follows from the inequalities above that these measures are always between 0 and 1, are 0 if and only if $X$, $Y$ are independent and are 1 if and only if $Y$ is a function of $X$, $X$ is a function of $Y$, each is a function of the other respectively. *D. Blackwell*

**Belevitch, V. Théorie de l'information et statistique linguistique.** Acad. Roy. Belg. Bull. Cl. Sci. (5) **42** (1956), 419–436.

By introducing Shannon's concept of information and the notion of cost attached to an element of a code, the author discusses how to maximize the mean information per unit cost. The maximization is used to explain the empirical rank-frequency relationship for English words when ranked by frequency of occurrence. Applications to speech-sounds and to letters in language using the roman alphabet are discussed. *D. V. Lindley.*

**Kendall, M. G.; and Lawley, D. N. The principles of factor analysis.** J. Roy. Statist. Soc. Ser. A. **119** (1956), 83–84.

The authors attempt to clarify the distinction between principal component analysis and factor analysis, a point on which considerable confusion exists. They also succintly review the general procedure in a factor analysis. *C. C. Craig* (Ann Arbor, Mich.).

**Watson, G. S. On the joint distribution of the circular serial correlation coefficients.** Biometrika **43** (1956), 161–168.

An exact formula for the joint distribution is derived for the case of a Gaussian process with zero autocorrelations and nonzero means. Method: Characteristic functions and the latent root representation of the serial coefficients. An approximation is obtained and compared with more precise results by Jenkins [Biometrika **41** (1955), 405–419; MR **16**, 605] and Daniels [see the following reviews]. *H. Wold* (Uppsala).

**Daniels, H. E. The approximate distribution of serial correlation coefficients.** Biometrika **43** (1956), 169–185.

The author applies his adaptation of the method of steepest descents [Ann. Math. Statist. **25** (1954), 631–650; MR **16**, 603] to improve upon earlier results, especially with regard to the order of approximation, and to treat more general cases. Distributions are derived for the first serial coefficient in a Markov process (two cases: circular and noncircular definition) and for the joint distribution of the first $m$ coefficients (two cases: ordinary and partial coefficients) in a circular autoregressive process of order $m$. Throughout the two cases of known and unknown mean are covered; all processes are stationary and Gaussian. The circular definition is easier to deal with, but he stresses that it is artificial and can only be justified if the ensuing results are not substantially affected by the noncircularity. A typical result is that with a Markov process Leipnik's formula for the circular case [ibid. **18** (1947), 80–87; MR **8**, 476] needs only a change in the degrees of freedom to apply in the noncircular case, with an error of order $O(T^{-3/2})$, where $T$ is the number of observations.

*H. Wold* (Uppsala).

**Jenkins, G. M. Tests of hypotheses in the linear autoregressive model. II. Null distributions for higher order schemes: non-null distributions.** Biometrika **43** (1956), 186–199.

[For part I see Biometrika **41** (1954), 405–449; MR **16**, 605.] Independently, part of the results of Daniels [see the preceding review], are proved by the use of moments based on the "smoothed" distribution of the serial covariances. Applications to hypothesis testing, a partial serial coefficient $r_k$ being known to form the likelihood ratio criterion for testing an autoregressive process of order $k-1$ against the hypothesis that the process is of order $k$.

The author criticises previous workers for fitting a model before testing it, but of course in constructing a ratio of maximised likelihoods one implicitly fits the two models under consideration. *H. Wold* (Uppsala).

**See also:** Hartley, p. 74; Migdal and Polietkov-Nikoladze, p. 98.

# PHYSICAL APPLICATIONS

## *Mechanics of Particles and Systems*

**Sbrana, Francesco. Sulle condizioni sufficienti per l'equilibrio di un sistema materiale, dedotte mediante il principio dei lavori virtuali.** Boll. Un. Mat. Ital. (3) **11** (1956), 123–125.

This note gives a simplification of the previously known proof of the sufficiency of the principle of virtual work for the equilibrium of a mechanical system. *L. A. MacColl.*

**Günther, Wilhelm. Spannungsfunktionen und Verträglichkeitsbedingungen der Kontinuumsmechanik.** Abh. Braunschweig. Wiss. Ges. **6** (1954), 207–219.

The author elaborates on considerations advanced

earlier by H. Schaefer [Z. Angew. Math. Mech. 33 (1953), 356–362; MR 15, 482] concerning the stress function of a continuous medium in equilibrium without body forces. This tensor can be used to define a non-Euclidean metric in which the stress tensor plays the same role as the energy-momentum tensor does in Einstein's theory of relativity. The author shows that the stress components can be interpreted as the Gaussian curvatures of appropriate geodesic surfaces. Furthermore, he derives formulae expressing the force and the moment exerted on a portion of the surface of the body as contour integrals involving the stress functions. *W. Noll* (Pittsburgh, Pa.).

**Wunderlich, W. Zur angenäherten Geradführung durch symmetrische Gelenkvierecke.** Z. Angew. Math. Mech. 36 (1956), 103–110. (English, French and Russian summaries)

In a symmetric convex plane four-bar linkage of base $A_1A_2$ (length $2a$, center $O$) and coupler $B_1B_2$ (length $2b<2a$, center $D$), $DC$ is an oriented segment of length $c$ perpendicular to $B_1B_2$ and pointing toward the base. In the central position the length of $OD$ is $d$. If an interval $\Gamma$ of the path of $C$ is to be Čebyšev's best approximation to a straight line segment $\Gamma_0$ through the central position $C_0$ of $C$, $\Gamma_0$ must be a triple tangent to $\Gamma$ at $C_0$.

By introducing the variable $u=\cot\theta$, where $\theta$ is the absolute angle between the cranks $A_1B_1$ and $A_2B_2$, the author makes all the equations (pertinent to this approximation) rational in $a$, $b$, $c$, $d$, $u$; he proves that $[(a-b)c+bd]^2=4(a+b)(2ab-cd)b$ is necessary and sufficient for the approximation, determines the values $u_0$, $u_2$ of $u$ for the contact points of $\Gamma$ and $\Gamma_0$ as fractions rational in $a$, $b$, $c$, $d$, and expresses the last four quantities as polynomials in $u_0$, $u_2$.

Among the numerous details of the paper are included an expression for the upper bound of the orthogonal deviation of $\Gamma$ from $\Gamma_0$, several nomograms, and a discussion of the case in which one part of the path of $C$ is the best approximation to $\Gamma_0$, while the other is the best approximation to a straight line segment $\Gamma_1$ parallel to $\Gamma_0$. The whole path consists then of three loops. There is also a mention of the case when the cranks cross.

*A. W. Wundheiler* (Chicago, Ill.).

**Rodov, A. M. On the exposition of the Leibnitz-Lagrange variational principle.** Belorussk. Gos. Univ. Uč. Zap. Ser. Fiz.-Mat. 15 (1953), 18–21. (Russian)

A modified version of the proof of the theorem that, for a system of free particles, the Newtonian motion yields a stationary value of the integral $\int_{t_0}^{t_1} T\,dt$ ($T$ the kinetic energy) if the competing motions have the same configurations at $t_0$ and $t_1$ and satisfy the energy integral with the same energy constant. *A. W. Wundheiler.*

**Hübner, Gerhard; und Lübcke, Ernst. Zur Einwirkung von periodischen, räumlich verteilten Kräften auf die Schwingungen mechanischer Schwingungsgebilde.** Z. Naturf. 11a (1956), 492–498.

General vibrating systems are considered with free vibrations according to the differential equation

$$M[y]=N\left[\frac{\partial^2 y}{\partial t^2}+2b\frac{\partial y}{\partial t}\right],$$

where $y$ is the coordinate determining the elongation in the vibration considered, $x$ is the coordinate determining a point of the system, $M$ and $N$ are functions involving derivatives of their arguments with regard to $x$, up to the $m$th order, $m=2$ in most of the examples mentioned by the author (strings, bars, rings, circular arcs) in some of them $m=4$. There is a frictional resistance proportional to velocity, rendered by the term with $b$, $b$ is supposed independent of $x$ and $t$.

The system is subject to a periodic force $P=P(x)\sin\omega t$. The forced vibrations are then given by a differential equation

$$M[y]=N\left[\frac{\partial^2 y}{\partial t^2}+2b\frac{\partial y}{\partial t}\right]-p(x)\sin\omega t,$$

$2m$ homogeneous linear boundary conditions and 2 initial conditions. $p(x)$ depends on $P(x)$ but is not necessarily identical with it.

The solution is effectuated by expanding $y$ in a series of eigenfunctions $u_i(x)$ and $p(x)$ in a series of functions $N[u_i(x)]$ with coefficients $p_i$.

The most interesting result relates to resonance. For resonance two conditions are necessary: 1) The well-known condition that the frequency of the external force is equal to one of the eigenvalues of the system and 2) that the corresponding $p_i$ is not zero. *H. Bremekamp.*

**Aggarwal, S. P. Internal ballistics for power law of burning with most general form function.** Proc. Nat. Inst. Sci. India. Part A. 21 (1955), 428–435 (1956).

The author states "All the present theories of internal ballistics of guns assume a linear law of burning of the propellants as it renders the mathematical treatment of the ballistic problem comparatively easy. Since the experiments showed that many propellants burn according to the power law of burning, Clemmow in 1928 gave numerical solution of the equations of internal ballistics for power law of burning, assuming simple quadratic form function. Now in the present-day advancement of the ballistic theories the need of a better form function has arisen." The author considers a general analytic form, but by the fourth page settles on a cubic form function, and later on a simpler quadratic expression. The symbols for physical variables are not explained. No numerical results are obtained. {This reviewer remarks that under his direction tables for Interior Ballistics were prepared for the U. S. Army as early as 1920 using a geometrical, nonanalytic burning law (form function) for the U.S. standard 7-perforate smokeless powder artillery grain, made up of a quadratic progressive law lasting from ignition to completion of web-burning, continued by a degressive law for burning of the splinters up to "all-burnt". [For some critical comments on the general objective of this paper, consult J. Corner, Theory of the interior ballistics of guns, Wiley, New York, 1950, pp. 33 ff.; MR 12, 213.]} *A. A. Bennett* (Providence, R.I.).

★ **Bilimović, Anton. Dinamika čvrstog tela. [Dynamics of a rigid body.]** Srpska Akademiya Nauka. Posebna Izdanja. Knj. 248. Mat. Inst. Knj. 2. Beograd, 1955. xi+176 pp.

During the author's 35 years of teaching theoretical mechanics at the University of Beograd he has also been active in research, and, in particular, in the field of dynamics of rigid bodies. This fact has its impact on the present book which covers the standard material, using the elements of vector and tensor calculus and theory of matrices. A chronological list of the author's 80 papers on vector analysis and rational mechanics follows the preface of the book.

The chapter headings are the following. I. Quantity of

motion, moment of momentum and the force vive of a rigid body. II. Differential equations of motion of a rigid body. III. Motion about a fixed axis. IV. Motion about a fixed point. V. Plane motion. VI. Gyroscope. VII. Statics of rigid bodies. VIII. Impact. The book concludes with a short history of rational mechanics and a common authors and subject index. *E. Leimanis.*

**Kaeppeler, H. J. Über eine simultane analytische Integration der Bewegungsgleichungen eines geflügelten Gerätes im Überschallgleitflug.** Astronaut. Acta 1 (1955), fasc. 4, 166–170.

The differential equations of motion of a winged vehicle in non-powered supersonic glide are integrated analytically, it being assumed that the angle of inclination of the trajectory tangent and the lift-to-drag ratio remain constant. The variation of gravity with altitude and arbitrary atmospheric data are taken into account. Formulae are given for the calculation of the velocity and time as functions of altitude. The results are considered to be of practical value in engineering. *L. A. MacColl* (New York, N.Y.).

**Lawden, D. F. Optimal transfer between circular orbits about two planets.** Astronaut. Acta 1 (1955), fasc. 2, 89–99.

The author examines the problem of minimal fuel expenditure for a rocket starting in the gravitational field of one planet and ending in the gravitational field of a second planet, the planets having coplanar elliptical orbits. Relative minima only are obtained initially, and these discrete results must then be compared to identify the absolute minimum. The optimal trajectory will consist of a number of arcs of null thrust, at the junctions of which impulsive thrusts are applied. The transfer orbit is hyperbolic towards the planet of arrival. The mathematical discussion, while complicated, applies standard techniques of rational dynamics. *A. A. Bennett.*

**See also:** Saltykow, p. 44; Heinrich and Desoyer, p. 91.

## *Statistical Mechanics*

**Grosjean, C. C. Solution of a non-isotropic random flight problem in the case of a non-isotropic point source.** Nuovo Cimento (9) 11 (1954), 11–40.

The following problem is treated: A point source emitting neutrons with an angular dependence $(1+A\cos\alpha)$ ($\alpha$ is the polar angle with respect to a chosen $z$-axis) is embedded in a medium which scatters them with a phase function $(1+3b\cos\gamma)$ ($\gamma$ is the angle of scattering); and the probability distribution of mean free paths, $l$, between successive scatterings is governed by $f(l)$. Given all these, what is the velocity and space distribution of the neutrons? The formal solution for this problem is obtained by methods which are extensions of those by the author in earlier papers [Verh. Kon. Vlaamse Acad. Wetensch. Lett. Schone Kunst. België 13 (1951), no 36; 14 (1952), no. 13; MR 15, 970]. Thus he introduces the functions $\phi_n(r, \alpha; \xi, \nu)$ and $\psi_n(r, \alpha; \xi, \nu)$ such that

$$\phi_n dr \frac{\sin\alpha\, d\alpha}{2}\frac{\sin\xi\, d\xi\, d\nu}{4\pi} \text{ and } \psi_n\, dr \frac{\sin\alpha\, d\alpha}{2}\frac{\sin\xi\, d\xi\, d\nu}{4\pi}$$

are the elementary probabilities that a particle emitted by the source would undergo the $n$th scattering in the shell $\{(r, r+dr), (\alpha, \alpha+d\alpha)\}$ and that it would be moving, respectively, continue in its trajectory, in the element of solid angle $\{(\xi, \xi+d\xi), (\nu, \nu+d\nu)\}$ with respect to the local frame of reference at the point of scattering. The functions $\phi_n$ and $\psi_n$ are related by

$$\psi_n(r, \alpha; \xi, \nu) = \frac{1}{4\pi}\iint \phi_n(r, \alpha; \xi', \nu')(1+3b\cos\gamma)\sin\xi' d\xi' d\nu',$$

where $\cos\gamma$ is the angle between the directions $(\xi, \nu)$ and $(\xi', \nu')$. Expanding the functions $\phi_n$ in spherical harmonics in the manner

$$\phi_n(r, \alpha; \xi, \nu) = \sum_{k=0}^{\infty}\sum_{m=0}^{k} \phi_n^{(k,m)}(r, \alpha) P_k^m(\cos\xi)\cos m\nu$$

the relation

$$\psi_n(r, \alpha; \xi, \nu) = \phi_n^{(0,0)}(r, \alpha) + b\{\phi_n^{(0,1)}(r, \alpha)\cos\xi + \phi_n^{(1,1)}(r, \alpha)\sin\xi\cos\nu\},$$

is deduced. Finally, from the definition of the functions $\phi_n$ and $\psi_n$ the recurrence relation

$$(*)\quad \phi_n(r, \alpha; \xi, \nu) = r^2\int_0^\infty \frac{\psi_{n-1}(r', \alpha'; \theta, \zeta)}{r'^2} f(r'')dr''$$
$$= r^2\int_0^\infty \frac{\phi_{n-1}^{(0,0)}(r', \alpha')}{r'^2} f(r'')dr'' + br^2\int_0^\infty \frac{\phi_{n-1}^{(1,0)}(r', \alpha')}{r'^2}\cos\theta f(r'')dr'' + br^2\int_0^\infty \frac{\phi_{n-1}^{(1,1)}(r', \alpha')}{r'^2}\sin\theta\cos\zeta\, f(r'')dr'',$$

is deduced where $\mathbf{r}=\mathbf{r}'+\mathbf{r}''$, $\theta$ is the angle between the directions of $\mathbf{r}'$ and $\mathbf{r}''$ and $\zeta$ is the azimuth angle which goes with $\theta$. In particular for the case $n=2$, we have

$$\phi_2(r, \alpha; \xi, \nu) = r^2\int_0^\infty \frac{f(r')f(r'')}{r'^2}(1+A\cos\alpha')dr'' + 3br^2\int_0^\infty \frac{f(r')f(r'')}{r'^2}(1+A\cos\alpha')\cos\theta\, dr''.$$

By expressing $f(r)/r$ in the form

$$\frac{f(r)}{r} = \int_0^\infty \mathfrak{F}(u)\sin ru\, du,$$

the various integrals in (*) are expressed in terms of the functions

$$F^{(k,m)}(u) = \sqrt{\left(\frac{\pi}{2}\right)}\int_0^\infty f(y)\left[i^m P_m\left(\frac{d}{idz}\right)\frac{J_{k+\frac{1}{2}}(z)}{\sqrt{z}}\right]_{z=yu} dy.$$

In this manner the complete formal solution is obtained. In the manipulation, extensive use is made of the author's addition theorem [Meded. Kon. Vlaamse Acad. Wetensch. Lett. Schone Kunst. België 14, no. 13 (1952) 33 pp.; MR 15, 970]:

$$\frac{j_{m+\frac{1}{2}}(r'u)}{r'u}P_m(\cos\theta) = \sum_{k=0}^{\infty}(2k+1)P_k(\cos\xi)\frac{j_{k+\frac{1}{2}}(ru)}{ru} \times \left[i^m P_m\left(\frac{d}{idz}\right)\frac{j_{k+\frac{1}{2}}(z)}{z}\right]_{z=r''w}$$

where $j_{k+\frac{1}{2}}(x) = (\pi x/2)^{\frac{1}{2}} J_{k+\frac{1}{2}}(x)$ and $\mathbf{r}$, $\mathbf{r}'$ and $\mathbf{r}''$ are related as in (*). *S. Chandrasekhar* (Williams Bay, Wis.).

**Stupočenko, E. V. On the distribution of kinetic energy in a "single-component" system with particle sources.** Vestnik Moskov. Univ. 8 (1953), no. 8, 57–71. (Russian)

The problem is considered of determining the velocity

distribution function in a gas consisting of particles, all of one kind, and containing sources of these particles uniformly distributed throughout the volume. The Boltzmann equation for the distribution function is set up, and a solution is sought by means of an expansion in powers of a parameter associated with the strength of the sources. For the case of monochromatic sources an approximate solution is obtained. *N. Rosen* (Haifa).

**Klein, Martin J. Entropy and the Ehrenfest urn model.** Physica 22 (1956), 569–575.

The Ehrenfest urn model is used to illustrate the difference between the Boltzmann and the Gibbs definitions of entropy. The former is a stochastic variable, a function of the occupation number and shows fluctuations for all times. The Gibbs entropy is a functional of the probability distribution and is shown to increase monotonically in (discrete) time. *G. Newell* (Providence, R.I.).

**Mayer, Joseph E. Structure of simple fluids.** Bull. Amer. Math. Soc. 62 (1956), 332–346.

This, the twenty-ninth Josiah Willard Gibbs Lecture presented in December 1955, gives a very general review of the present status of attempts to describe the equilibrium properties of dense gases and liquids using statistical mechanics. The discussion is confined mainly to describing the method of attack and the difficulties encountered in the virial expansion, cell model and integral equation methods. *G. Newell.*

**Watson, K. M. Applications of scattering theory to quantum statistical mechanics.** Phys. Rev. (2) 103 (1956), 489–498.

This is a paper applying the Brueckner-Watson methods of attacking the many-body problem to quantum statistical mechanics. By using the formal correspondence between $1/kT$ in statistical mechanics and $it/\hbar$ in the Schrödinger equation, the partition function is expressed formally as the solution of a many-body scattering problem. The Brueckner-Watson formalism attacks this problem by a method which in principle starts from the exact solution of the two-body problem and expands in successive "cluster" terms. A further section suggests the use of the "optical model" idea: the replacement of the scattering problem by an equivalent dispersive medium, as has been done for the nuclear problem by Brueckner et al. Few physical examples are studied, the emphasis being upon making the methods available.
*P. W. Anderson* (Murray Hill, N.J.).

**See also:** Bolton and Scoins, p. 72; Reid, p. 88; Patterson, p. 89.

## *Elasticity, Visco-elasticity, Plasticity*

**Nowacki, Witold. Some boundary problems of the theory of elasticity.** Arch. Mech. Stos. 7 (1955), 483–502. (Polish. Russian and English summaries)

The authors considers an elastic body in equilibrium under the influence of external loads, body forces, rise in temperature and initial stresses. Boundary conditions are imposed on arbitrary surfaces $\phi$ and $\Omega$, which are parts of the surface of the elastic body in problem. The problem consists of finding expressions for displacements, deformations and stresses. The author expresses displacements as integrals with Green functions which must satisfy boundary conditions. This leads to a system of three Fredholm integral equations of the first kind where forces on one of the boundaries are the unknown functions. The above can be extended to an elastic body with boundary conditions prescribed on $n$ surfaces leading to a system of $3n$ Fredholm integral equations. The author analyzes also an alternative where the unknown functions are displacement components. The methods presented by the author, although clear and compact are of such generality that they are difficult to apply to three dimensional problems except in special cases. Two-dimensional problems do not present such difficulties and the author's method offers certain advantages. In the second part of his paper the author shows applications to several problems in the theory of plates. *T. Leser.*

**Lee, E. H. Stress analysis in viscoelastic materials.** J. Appl. Phys. 27 (1956), 665–672.

Author reviews work done on the three main problems of stress analysis, namely measurement of material properties, determination of the corresponding stress-strain law and evaluation of stress in specific cases. Many references and given. *D. R. Bland* (London).

★**Prager, William. Discontinuous fields of plastic stress and flow.** Proceedings of the Second U. S. National Congress of Applied Mechanics, Ann Arbor, 1954, pp. 21–32. American Society of Mechanical Engineers, New York, 1955. $9.00.

This paper gives a general discussion of discontinuities in (generalized) stress and associated velocity fields for ideal-plastic, rigid solids. As an essential preliminary, terminology, both mathematical and physical in nature, is introduced to assist the classification of discontinuities at points, lines or surfaces in relation to the fundamental field equations. The paper then proceeds with a masterly review of practically all the various types of discontinuity yet recorded in the varied branches of the subject. This review is enlivened through reference to specific problems, too numerous to be detailed here, and a full bibliography is appended. The present discussion is concerned exclusively with the case when there is a single, continuously-differentiable yield function. Attention is drawn to the fact that no similar treatment is available for the more general and of course very important case when the yield criterion is expressed in terms of two or more such functions. Likewise, no general treatment of discontinuities is available for solids exhibiting more complicated types of mechanical behaviour.

{The reviewer adds that the importance of the role of discontinuities in the mathematical theory of plasticity can scarcely be over-emphasized. In particular, the study of discontinuities leads naturally to the elucidation of the mathematical nature of the governing equations that apply in any special case. In such matters, as the author himself emphasizes, it is of course highly necessary to keep purely mathematical questions distinct from purely physical ones.} *H. G. Hopkins* (Sevenoaks).

**Pearson, Carl E. General theory of elastic stability.** Quart. Appl. Math. 14 (1956), 133–144.

Some general topics in the elastic stability of conservative systems are discussed. It is shown that, given an arbitrary elastic body in equilibrium under certain loading, the necessary and sufficient condition for instability is that, for some allowable virtual displacement, a positive discrepancy should exist between the work done

by the loading and the increase in internal energy. In calculating the latter, the exact stress-strain relationship of non-linear elasticity due to Murnaghan [Finite deformations of an elastic solid, Wiley, New York, 1951; MR 13, 600] is used. It is shown that, on the same general basis, the more conventional "adjacent equilibrium position" or "neutral equilibrium state" criterion will lead, with proper formulation, to the same eigenvalue problem (for conservative systems) as does the minimization of the instability condition in the energy approach. The analytical expression for the instability condition is also shown to be capable of modification to include both dead loading and pressure loading effects. The buckling of a cylindrical shell under external pressure is taken as an example of the sensitivity of certain stability problems to the character of the loading. In addition to presenting the author's own contribution, this paper contains a worthwhile summary and critical discussion of prior work in this field. *W. Nachbar* (Van Nuys, Calif.).

**Rivlin, R. S. Stress-relaxation in incompressible elastic materials at constant deformation.** Quart. Appl. Math. **14** (1956), 83–89.

The "stress-relaxing elastic materials" of the author are defined by constitutive equations in which the stress depends not only on the displacement gradients with respect to an initial state but also on the time. Except for this time dependence, the equations reduce to the same form as do those of the classical theory of finite deformations. The author points out that many of the well-known special results of this latter theory can be transferred with little difficulty to his new theory. In particular he considers simple extension and simultaneous simple extension and torsion. {The reviewer objects to time dependent constitutive equations because of the unnatural dependence on the choice of the initial instant. However, he feels that the author's equations should and in fact do appear as consequences of genuine constitutive equations; cf. Noll, J. Rational Mech. Anal. **4** (1955), 3–81, p. 19; MR **16**, 764.} *W. Noll* (Pittsburgh, Pa.).

**Voelz, Kurt. Die Theorie der inneren Dämpfung schwingender fester Körper.** Abh. Braunschweig. Wiss. Ges. **6** (1954), 126–165.

This is an expository paper on existing one-dimensional phenomenological as well as molecular theories of viscoelasticity and relaxation. *W. Noll* (Pittsburgh, Pa.).

**Kröner, Ekkehart. Die Spannungsfunktionen der dreidimensionalen isotropen Elastizitätstheorie.** Z. Physik **139** (1954), 175–188.

The author shows that the stress tensor $\sigma$ in an isotropic linearly elastic body can be represented in terms of a harmonic tensor $\psi$ of stress functions in the form

$$(*)\qquad \sigma=\ln k\left(\psi+\frac{\frac{1}{2}m}{m-1}\mathbf{r}\nabla\cdots\psi I,\right)$$

where dyadic notation is used, and where $\ln k\,\varphi=\nabla\times\varphi\times\nabla$, $m$=Poisson's modulus, $\mathbf{r}$=radius vector, $I$=unit tensor. {Reviewer's remark: In standard tensor notation (*) reads

$$\sigma^{ik}=\psi_{jl}+\frac{\frac{1}{2}m}{m-1}r_p\psi^{pq}\delta_{jl})_{,r,s}\varepsilon^{ijr}\varepsilon^{kls}.\}$$

Body forces and residual stresses are assumed to be absent. In any Cartesian coordinate system, $\psi$ can be chosen such that only the three diagonal components of $\psi$ are different from zero. Then (*) is a representation of the stress tensor by three harmonic functions. It is similiar and related to the representation of the strain tensor by the Papkovitch-Neuber displacement potentials .[For a recent review of this subject cf. K. Marguerre, Z. Angew. Math. Mech. **35** (1955), 242–263; MR **16**, 1068.] *W. Noll* (Pittsburgh, Pa.).

**Dorn, W. S.; and Schild, A. A converse to the virtual work theorem for deformable solids.** Quart. Appl. Math. **14** (1956), 209–213.

In three dimensions, let $u_i$ be any vector field defined on the boundary $S$ of a region $V$. Let $\varepsilon_{ij}$ be any symmetric tensor defined throughout $V$. The authors show that, assuming sufficient continuity, if

$$\int\sigma_{ij}\varepsilon_{ij}\,dV=\oint u_i\sigma_{ij}n_j\,dS$$

for all symmetric tensors $\sigma_{ij}$ such that $\sigma_{ij,j}=0$, then there exists a vector field $U_i$ such that $U_i=u_i$ on $S$, $\varepsilon_{ij}=\frac{1}{2}(U_{i,j}+U_{j,i})$ in $V$. Here $n_i$ is the unit normal to $S$. References to related work are given. *J. L. Ericksen.*

**Bishop, J. F. W.; Green, A. P.; and Hill, R. A note on the deformable region in a rigid-plastic body.** J. Mech. Phys. Solids **4** (1956), 256–258.

A complete solution for an ideal, work-hardening rigid-plastic material whose yield function and plastic potential are identical consists of an equilibrium stress distribution satisfying the stress boundary conditions and nowhere violating the yield criterion, together with one more mode of deformation satisfying the velocity boundary conditions and compatible with the stress distribution. Using a uniqueness theorem on stresses given by Hill [Phil. Mag. (7) **42** (1951), 868–875; MR 13, 185], the authors show that any region of the stress field in a known complete solution which is shown to be necessarily rigid for that field must be rigid in all complete solutions. They then conclude that to determine the deformable region only one complete solution need be considered. A method is given for such a determination when the velocity equations are hyperbolic with distinct characteristics. It is illustrated in an indentation problem. *G. H. Handelman* (Troy, N.Y.).

**Hill, R. On the problem of uniqueness in the theory of a rigid-plastic solid. I.** J. Mech. Phys. Solids **4** (1956), 247–255.

A solid which is work-hardening and rigid-plastic with an identical yield function and plastic potential at each point is considered. The author has previously shown [Phil. Mag. (7) **42** (1951), 868–875; MR **13**, 185] that if the surface tractions are given over part of the surface as well as the velocity components over the remainder, the state of stress in the region which can deform is uniquely determined. Bishop, Green, and Hill [see the preceding review] have given methods for determining this zone. The author shows that if there is more than one mode of deformation compatible with the boundary conditions, the physically possible mode is singled out by the additional requirement that there must exist an equilibrium distribution of stress-rate compatible with the implied rate of hardening everywhere in the body and with the given traction rate on the surface. It is also shown that if the velocities are specified on part of the surface, the stress-rate components individually are not uniquely determined unless the accelerations are also specified.

These uniqueness theorems are formally similar to those previously given by the author for elastic-plastic materials [Mathematical theory of plasticity, Oxford, 1950, pp. 53–58; MR 12, 303] but the interpretation and conclusions are fundamentally different. In addition, several extremum principles are proved and these in turn can be used to obtain upper and lower bounds on certain physically significant quantities.

Since no uniqueness theorem can be established for a nonhardening solid, the author has proposed an additional physical hypothesis. Consider a work — hardening solid with a work — hardening function $\mu h$ ($\mu$ a parameter). The appropriate solution for the nonhardening solid is found by considering the unique sequence of actual modes of work-hardening solids found as $\mu$ approaches zero, provided the limit exists.

Possible discontinuities in the velocity field are also discussed. *G. H. Handelman* (Troy, N.Y.).

★ **Sanders, J. L., Jr. Plastic stress-strain relations based on linear loading functions.** Proceedings of the Second U. S. National Congress of Applied Mechanics, Ann Arbor, 1954, pp. 455–460. American Society of Mechanical Engineers, New York, 1955. $9.00.

Recently a generalization of incremental theories of plasticity was made in which the concept of relating the strain increment to the gradient of loading function is retained even when the yield surface has corners [Koiter, Quart. Appl. Math. 11 (1953), 350–354; MR 15, 583]. This was accomplished by the simple device of introducing more than one loading function.

The present paper discusses the possibilities of using the plane loading surface as the fundamental building block in the construction of stress-strain relations for work-hardening solids. It is shown that in the case of independently acting loading planes the yield surface develops corners as loading proceeds. On the other hand the author remarks that the consideration of interdependent plane loading surfaces may enable one to construct the stress-strain relations for materials exhibiting Bauschinger and cross effects. *E. T. Onat.*

**Lee, Lawrence H. N. Non-uniform torsion of tapered *I*-beams.** J. Franklin Inst. 262 (1956), 37–44.

In dieser Arbeit wird eine rechnerische Untersuchung von verjüngten Trägern mit Doppeltflanschen bei ungleichmässiger Verdrehung durchgeführt. Das Resultat der Berechnung stimmt für diejenige Klasse von Profilen, bei denen das Verhältnis der Steghöhe zur Dicke des Flanches gross ist, mit Versuchen gut überein. Die Wirkung der Formänderung des Steges, die für Balken von konstantem Querschnitt von Goodier und Barton [J. Appl. Mech. 11 (1944), A-35–A-40] in Betracht gezogen wurde, hat sich für das erwähnte Verhältnis als relativ bedeutungslos herausgestellt und wurde daher für den vorliegenden Fall ausser Acht gelassen. Die erhaltenen Ergebnisse sind an numerischen Beispielen erläutert und in Kurven dargestellt. *R. Gran Olsson.*

**Gerard, George. Torsional instability of hinged flanges stiffened by lips and bulbs.** NACA Tech. Note no. 3757 (1956), 12 pp.

**Dol'berg, M. D. On the longitudinal bending of multispan rods on rigid supports.** Har'kov. Gos. Univ. Uč. Zap. 34=Zap. Mat. Otd. Fiz.-Mat. Fak. i Har'kov. Mat. Obšč. (4) 22 (1950), 127–143 (1951). (Russian)

There used to be an opinion that the buckling of an axially compressed strut does not depend on where the compressive forces are applied and that the first mode must be without nodes. O. Blumenthal [Z. Angew. Math. Mech. 17 (1937), 232–244] proved that if the compressive axial loads are applied to certain parts of the beam then the first mode will have one node, and he also derived the necessary condition for a one-span beam to have the first mode without nodes.

Following Blumenthal, the author analyzes problems of stability of a multispan beam on rigid supports as follows: 1. The number of modes corresponding to the $n$th critical load; 2. the number of nodes in a given $n$th mode; 3. how an additional constraint (the beam is prevented from moving on a support in axial direction) influences the critical load.

The author's analysis is based on an integral equation given by F. Trefftz [ibid. 3 (1923), 272–275], which is equivalent to the fourth order stability differential equation. *T. Leser* (Aberdeen, Md.).

**Bölcskei, E. Limit load capacity of the compression bar.** Acta Tech. Acad. Sci. Hungar. 15 (1956), 19–35. (Russian, French and German summaries)

The limit state of strength of a pin-supported bar subject to axial forces applies eccentrically at its ends is considered. The bar is idealized to two outer members, which carry the load, connected by structural members that take no load. A cross-section is said to arrive at the limit state if the crushing strain arises on the concave side or the rupture strain on the concave side. An elastic-plastic stress-strain law for uniaxial stress and strain is used. The cross-sections are assumed to remain plane during the deformation. The author calculates the values of the eccentricity and the axial force for which the limiting state can occur. Three examples based on different stress-strain laws are given. *G. H. Handelman.*

**Seide, Paul. Elasto-plastic bending of beams on elastic foundations.** J. Aero. Sci. 23 (1956), 563–570.

Author assumes bilinear stress-strain curve and linearises problem by replacing moment-curvature diagram with its linear asymptotes. The accuracy of this assumption is tested on the simple example of an infinite beam under a single concentrated load, for which the nonlinear fourth order differential is solved numerically and this solution compared with the linearised one.

The agreement obtained is satisfactory and experimental results also seem to justify the approach.

*W. Freiberger* (Providence, R.I.).

**Krall, Giulio. Sul problema centrale della dinamica sui ponti. I.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 19 (1955), 373–381 (1956).

This note concerns the general problem of the transverse deflection of an elastic beam due to a travelling load. This problem is particularly important in connexion with the design of railway bridges. Following the review of previous work, a brief discussion is given of some special problems.

*H. G. Hopkins* (Sevenoaks).

**Shuleshko, P. Buckling of rectangular plates with one unsupported edge, compressed by forces distributed along its edges.** J. Roy. Aero. Soc. 60 (1956), 488–489.

The elastic stability of rectangular plates compressed in two perpendicular directions by forces uniformly distributed along the edges is considered. Edges $x=0$ and $x=a$ are simply supported, edge $y=0$ is free and edge

$y=b$ is rigidly built in. Using a method described in a previous paper [J. Inst. Engrs. Australia **25** (1953), No. 4-5] after satisfying the boundary conditions at the edges of the plate, a transcendental equation is established, from which the eigenvalues may be obtained.

*R. Gran Olsson* (Trondheim).

**Özden, K. Biegung dünner Platten und Variationssätze bei einem nichtlinearen Elastizitätsgesetz.** Ing.-Arch. **24** (1956), 133–147.

The author discusses a nonlinear elasticity theory, according to which the stress is a particular type of nonlinear function of infinitesimal strain, derivable from a strain energy. Approximate forms of the equations for thin plates are obtained. Approximate solutions are obtained by variational and power series methods and compared with predictions of linear elasticity.

*J. L. Ericksen* (Washington, D.C.).

**Aramanovič, I. G. On stress distribution in an elastic half-plane weakened by a reinforced circular opening.** Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 372–375. (Russian)

The problem of a semi-infinite space with a nonreinforced circular cylindrical hole inside, was solved by G. Jeffery [Philos. Trans. Roy. Soc. London. Ser. A. **221**, (1920), 265–293] and the problem of a semi-infinite space with a reinforced circular cylindrical hole at an infinite distance from the bounding plane was considered by S. G. Mikhlin [Prikl. Mat. Meh. **2** (1934), 82–90].

The author of this paper solves the problem of a semi-infinite space with a reinforced circular hole at a finite distance from the bounding plane using the Muskelishvili's method of complex stress functions. *T. Leser.*

**Morgan, Antony J. A. Stress distributions within solids bounded by one or two cones.** Z. Angew. Math. Phys. **7** (1956), 130–145.

For infinite or semi-infinite solids bounded by one or two right circular cones, the author obtains formal solutions to general classes of stress boundary value problems, using the equations of linear elasticity for isotropic materials. Formulae for the stress components are derived using the Mellin transform. Corresponding displacement fields are not calculated. *J. L. Ericksen.*

**Čhikvadze, G. M. Skew bending by a couple of composite prismatic bars.** Soobšč. Akad. Nauk Gruzin. SSR **16** (1955), 425–430. (Russian)

Skew bending by a couple of a homogeneous beam in non-linear theory of elasticity was solved by A. I. Pojalostin and P. M. Riz [Prikl. Mat. Meh. **6** (1942), 375–380; MR **5**, 26].

The author applies the non-linear theory to a cylindrical beam consisting of several parallel prismatic bars of different material placed in an elastic medium which separates one from another. The elastic medium is bounded by a cylindrical surface whose elements are parallel to the axes of all component bars. The elastic medium together with the component bars from the composite beam. The author assumes: (a) the same Poisson ratio for all bars and for the elastic medium, (b) displacement vectors across dividing surfaces remain continuous, (c) internal forces on the opposite sides of a dividing surface are numerically equal but have opposite signs. The beam is loaded at an end by two couples whose moments are at a right angle to each other and are parallel to coordinate axes. The lateral surface of the beam is free of external loads. The author derives general formulas for displacements using methods outlined by N. I. Muskhelishvili [Some basic problems of the mathematical theory of elasticity, 3rd ed., Izdat. Akad. Nauk SSSR, Moscow-Leningrad; 1953; MR **11**, 626; **15**, 370] and by F. D. Murnaghan [Amer. J. Math. **59** (1937), 235–260].

*T. Leser* (Aberdeen, Md.).

**Miller, G. F.; and Musgrave, M. J. P. On the propagation of elastic waves in aeolotropic media. III. Media of cubic symmetry.** Proc. Roy. Soc. London. Ser. A. **236** (1956), 352–383.

[For parts I–II see same Proc. **226** (1954), 339–355, 356–366; MR **16**, 424, 425.] The authors give a detailed analysis of velocity, inverse and wave surfaces of media with cubic symmetry, using linear elasticity theory and experimental data for aluminum and nickel. Points of physical interest suggested by their analysis are noted as they arise and are briefly summarized. *J. L. Ericksen.*

**Davin, Marcel. Sur la vibration forcée d'un sol stratifié.** C. R. Acad. Sci. Paris **243** (1956), 352–354.

The forced vibration of a stratified elastic half-space is considered wherein the normal pressure loading on the plane boundary is periodic in time with fixed frequency but with an arbitrarily given distribution of amplitude over the plane. A solution is presented for the case of a half-space of two layers acted upon by single component of the Fourier representation for the amplitude.

*W. Nachbar* (Van Nuys, Calif.).

**Prager, William. Théorie générale des états limites d'équilibre.** J. Math. Pures Appl. (9) **34** (1955), 395–406.

The paper is concerned with the theorems of limit analysis. First of all the author remarks that the origins of the fundamental concepts of the limit analysis may be found in the methods of structural analysis that were employed before the advent of the theory of elasticity. The theorems of limit analysis are then stated and proved. The usual assumption that the yield surface has continuously turning tangent planes is not made in this paper. Therefore the proof of the theorems of limit analysis given in this paper is more general than the previous ones [Drucker, Greenberg and Prager, J. Appl. Mech. **18** (1951), 371–378; Quart. Appl. Math. **9** (1951), 381–389; MR **14**, 431; **13**, 603; Hill, Phil. Mag. (7) **42** (1951), 868–875; MR **13**, 185]. Finally it is indicated that these theorems may find useful application outside the field of theory of plasticity. *E. T. Onat* (Ankara).

**Berry, J. G.; and Naghdi, P. M. On the vibration of elastic bodies having time-dependent boundary conditions.** Quart. Appl. Math. **14** (1956), 43–50.

The class of vibration problems discussed in this paper differs from the usual ones in that some or all of the boundary conditions are non-homogeneous. The approach used here is to transform the problem into one with homogeneous boundary conditions by a change in variable; the problem is thus reduced to a related static problem, an associated free vibration problem, and a quadrature. This procedure is essentially the same one used by Mindlin and Goodman [J. Appl. Mech. **17** (1950), 377 380; MR **12**, 459] and Herrmann [ibid. **21** (1954), 221–224] in discussing beam and rod vibrations with time-dependent boundary conditions, only now in more general

terms. To illustrate the method, the problem of a simply supported beam driven by a step-moment at one end is solved according to the Timoshenko beam theory. The problem was previously solved by Leonard and Budiansky [NACA Tech.Note no. **2874** (1953)], using the method of Laplace transform.

*Y. Y. Yu* (Syracuse, N.Y.).

**Roesler, F. C. Glancing angle reflection of elastic waves from a free boundary.** Phil. Mag. (7) **46** (1955), 517–526.

An adaptation of the elementary wave-reflection method to treat the case of glancing incidence of a pressure pulse at a free boundary is proposed on heuristic grounds. Let $A_1$, $A_2$, $A_3$ be the amplitudes, respectively, of a uniform rectangular pressure pulse $P_1$ incident at angle $\alpha$, the reflected pulse $P_2$, and the reflected shear pulse $S$. It is shown the the ratios $\delta/\varepsilon$ and $\eta/\delta$, where $\delta=A_3/A_1$, $\varepsilon=(A_1+A_2)/A_1$, $\eta=\cos\alpha(A_1-A_2)$, remain finite as $\alpha\to\pi/2$ (glancing incidence); they therefore specify these amplitudes and (through the boundary conditions) the angle of reflection of $S$. This amounts to a description of $S$ in terms of a superposition of $P_1$ and $P_2$. The relation of this problem to that of diffraction of an elastic plane wave by a slot normal to the wave front is discussed.

*R. N. Goss* (San Diego, Calif.).

**Christie, D. G. Reflection of elastic waves from a free boundary.** Phil. Mag. (7) **46** (1955), 527–541.

The author presents experimental data on the problem of reflection from a free edge of a plane dilatational pulse traveling in a plate. Of particular interest are the results for glancing incidence, which tend to support the arguments of the paper reviewed above.

*R. N. Goss.*

**Mandel, Jean. Sur les corps viscoélastiques à comportement linéaire.** C. R. Acad. Sci. Paris **242** (1956), 2803–2805.

Substitution of the two coefficients of elasticity by time functions (memory functions) and application of a Laplace transform method gives the following general results: a) Propagation of discontinuities of strain in a viscoelastic medium is the same as in a purely elastic medium whose coefficients of elasticity are identical with the initial values of the memory functions. b) Sinusoidal forcing (strain or stress applied) gives a steady state in a finite medium and progressive damped waves in an infinite medium. c) Spacial distribution of free vibrations in the viscoelastic medium is identical with that in the elastic one.

*B. Gross* (Rio de Janeiro).

**Kheiralla, Ahmad Ali. A new theory of fatigue.** Naturwissenschaften **43** (1956), 321–322.

The problem of fatigue failure is compared to that of the random walk of a point within a circle. A formal solution is obtained by Kluyver's method [Watson, A treatise on the theory of Bessel functions, 2nd ed., Cambridge, 1944, p. 420; MR **6**, 64] in connection with the use of discontinuous integrals. The problem is further compared to the diffusion problem with the required Green's function given.

*Y. Y. Yu* (Syracuse, N.Y.).

**See also:** Kislicyn, p. 48.

## *Fluid Mechanics, Acoustics*

**Landweber, L. On a generalization of Taylor's virtual mass relation for Rankine bodies.** Quart. Appl. Math. **14** (1956), 51–56.

The author establishes certain relations between the components of the mass tensor of the body, the added mass tensor of the liquid and the singularity distributions, when a rigid body moves in a liquid otherwise at rest.

*L. M. Milne-Thomson* (Providence, R.I.).

**Chandrasekhar, S. Axisymmetric magnetic fields and fluid motions.** Astrophys. J. **124** (1956), 232–243.

This paper contains a systematic discussion of the hydromagnetic equations for axisymmetric fields in the case of an inviscid, incompressible fluid of finite electrical conductivity. The discussion is based on a decomposition of the magnetic and velocity fields into their poloidal and toroidal components. It includes a full treatment of the problem of boundary conditions.

For the limiting case of infinite electrical conductivity the author derives, by a simple but ingenious argument, a theorem that includes as special cases the law of isorotation, theorems on force-free fields, and conditions for hydrostatic equilibrium. He also derives a number of integral relations that are valid in the general case.

*D. Layzer* (Cambridge, Mass.).

**Chandrasekhar, S. Effect of internal motions on the decay of a magnetic field in a fluid conductor.** Astrophys. J. **124** (1956), 244–265.

The author applies the general theory of the paper reviewed above to the problem named in the title. On the assumption that the magnetic field decays as exp $(-\lambda^2 t)$, the velocity field is constant in time, and non-uniform rotation is absent, the differential equation for the toroidal magnetic field $T$ reduces to

$$\Delta_5 T+\lambda^2 T=\frac{1}{\varpi}\frac{\partial(T,\varpi^2 U)}{\partial(z,\varpi)},$$

where $\Delta_5$ is the Laplacian operator in five dimentions, $z$ and $\varpi$ are cylindrical coordinates, and $U$ is the poloidal velocity field. For a given function $U$, solutions of this equation that satisfy the boundary conditions exist only for a discrete set of values of $\lambda^2$, the largest of which determines the rate of decay.

The author classifies the modes of the magnetic and velocity fields according to a scheme based upon expansions of the fields in Gegenbauer polynomials [Chandrasekhar, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 1–5; MR **17**, 561]. To determine which velocity modes have the greatest effect on the decay, he formulates an equation for the decay of the magnetic energy:

$$-\frac{d}{dt}\mathfrak{T}(n,j;m,k)=C(n,j)+\beta I(n,j;m,k;T).$$

Here $\mathfrak{T}(n,j;m,k;T)$ denotes the energy of what is initially an $(n,j)$ toroidal magnetic mode; $I(n,j;m,k;T)$ is proportional to the rate at which energy is transferred from the magnetic field to the $(m,k)$ velocity mode; $\beta$ is the amplitude of the velocity field; and $C(n,j)$ is a constant. A similar equation holds for the energy of a poloidal mode. The author evaluates the transfer function numerically for the (0, 1) magnetic modes and the (1, 1), (1, 2), (1, 3) velocity modes. The (1, 1) velocity mode influences the decay of the toroidal and poloidal magnetic

modes in opposite directions, while the other two velocity modes influence the decay of both fields in the same direction. Whether the motions accelerate or arrest the decay in any particular case depends on the sign of $\beta$.

Finally, the author solves the above-mentioned eigenvalue problem numerically for several values of $\beta$ and for combinations of the (0, 1) magnetic modes with the (1, 1) and (1, 2) velocity modes. Applying his results to the problem of the earth's magnetic field, he concludes that reasonable motions, if they occur in the appropriate patterns, can lengthen the decay time from 17,000 years — the value it would have in the absence of motions — to 500,000 years — a value suggested by recent experimental work. *D. Layzer* (Cambridge, Mass.).

**Rikitake, Tsuneji. Magneto-hydrodynamic oscillations of finite amplitude of a conducting fluid sphere.** Bull. Earthquake Res. Inst. Tokyo 33 (1955), 583–592. (Japanese summary)

In this paper the author considers hydromagnetic oscillations in a fluid sphere about a static equilibrium state of zero magnetic field. In studying these oscillations terms quadratic in the velocities are ignored; but the nonlinear terms in the magnetic field arising from the Lorentz force $\mathbf{j}\times\mathbf{H}$ are retained. The analysis is incomplete (as the author admits); nevertheless, the suggestion is made that such oscillations might exist and be relevant to theories of the origin of the earth's magnetic field. {The mathematical problem is not clearly formulated. If one restricts oneself to poloidal magnetic fields and meridional motions with symmetry about an axis, then the equations governing the oscillations are

$$\text{(i)} \qquad -\varpi^3\frac{\partial P}{\partial t}=\frac{\partial(\varpi^2 P, \varpi^2 U)}{\partial(z, \varpi)},$$

$$\text{(ii)} \qquad \varpi\Delta_5\frac{\partial U}{\partial t}=\frac{\partial(\Delta_5 P, \varpi^2 P)}{\partial(z, \varpi)},$$

where $P$ and $U$ are two scalars characterizing the magnetic field and the motion respectively; $\varpi$ and $z$ define a system of cylindrical polar co-ordinates, and $\Delta_5$ is the axisymmetric Laplacian for five-dimensional space. (These equations are special cases of the general equations 33–36 given in the paper reviewed second above). If $P$ is assumed to be a fundamental poloidal mode, then $\Delta_5 P=-\alpha^2{}_{j,n-1}P$ where $\alpha_{j,n-1}$ is the $j$th zero of $J_{n+\frac{1}{2}}(x)$, and (ii) reduces to

$$\text{(iii)} \qquad \Delta_5\frac{\partial U}{\partial t}=-\alpha_{j,n-1}{}^2\frac{\partial P^2}{\partial z}.$$

The problem which the author seeks to solve is: Do equations (i) and (iii) allow oscillatory solutions?}
*S. Chandrasekhar* (Williams Bay, Wis.).

**Pao, Siciy C. On the effect of fluid motion on the initial decay of a magnetic field in a sphere.** Astrophys. J. 124 (1956), 266–271.

Following a method used by J. C. Adams [Proc. Roy. Soc. London. Ser. A. 27 (1878), 63–71] to evaluate the integral of the product of three Legendre polynomials, the author derives the following formula for the integral of the product of three Gegenbauer polynomials:

$$\int_{-1}^{+1} C_l{}^{3/2}(\mu)C_m{}^{3/2}(\mu)C_n{}^{3/2}(\mu)(1-\mu^2)d\mu=$$
$$\delta_{l,n+m-2r}\frac{2^2\Gamma(2r+2)\Gamma(2m+2-2r)\Gamma(2n+2-2r)}{\Gamma^2(r+1)\Gamma^2(m+1-r)\Gamma^2(n+1-r)}\times\frac{\Gamma(n+m+3-r)\Gamma(n+m+2-r)}{\Gamma^2(2n+2m+4-2r)}.$$

The transfer function $I(n, j; m, k)$ introduced by Chandrasekhar in the paper reviewed third above is proportional to such an integral. The 'selection rules' that $l$, $m$, $n$ must satisfy the triangular inequality and their sum must be even may therefore be interpreted physically in terms of the interactions between modes of the magnetic field and the velocity field. In an appendix, the author evaluates the integral

$$\int_{-1}^{1} C_l{}^\lambda C_m{}^\lambda C_n{}^\lambda(1-\mu^2)^{\lambda-\frac{1}{2}}d\mu$$

for arbitrary $\lambda$. *D. Layzer* (Cambridge, Mass.).

**Fadnis, B. S. Axisymmetric flow in perfect fluid. I. Motion about a spheroid and circular disc.** Bull. Calcutta Math. Soc. 47 (1955), 143–152.

The axisymmetric flow of a perfect fluid past spheroids is treated in the case when the fluid is in uniform rigid-body rotation far upstream. The corresponding case for the sphere was treated by Taylor [Proc. Roy. Soc. London. Ser. A. **102** (1922), 180–189], who found a non-uniqueness of solutions. Here, for spheroids, solutions are found in series form. The same ambiguity is encountered; i.e., there are infinitely many solutions satisfying the condition of vanishing normal component at the surface. A special solution can be found in which the tangential component also vanishes. The circular disc is considered as a special case. *W. R. Sears* (Ithaca, N.Y.).

**Chow, Tse-Sun. On an initial value problem for flow of a viscous incompressible fluid in an unbounded region.** J. Rational Mech. Anal. **5** (1956), 263–276.

The author investigates motions of an unbounded viscous incompressible fluid under the assumption that the velocity and pressure fields are spatially almost periodic; supposing

$$\mathbf{v}(\mathbf{x}, t)=\sum \mathbf{A}(\mathbf{r}, t)[\cos(\mathbf{r}\cdot\mathbf{x})+\sin(\mathbf{r}\cdot\mathbf{x})]$$

(where the wave number vector $\mathbf{r}$ is permitted to be negative as well as positive), he first deduces from the Navier-Stokes equations the conditions:

$$\partial\mathbf{A}(\mathbf{r}, t)/\partial t=\mathcal{F}\{\mathbf{A}(\mathbf{r}, t)\}-\nu r^2\mathbf{A}(\mathbf{r}, t),$$

where the term $\mathcal{F}$ expresses the interaction of the different velocity wave components; assuming that, for $t$ small enough, $\mathbf{A}(\mathbf{r}, t)$ can be developped in a Taylor series in $t$, the problem of the determination of $\mathbf{A}(\mathbf{r}, t)$, corresponding to a given initial almost periodic velocity field, is reduced to an integral equation; this equation is solved by an iteration procedure. *J. Kampé de Fériet* (Lille).

**Beran, Mark. A note on laminar axially symmetric jets.** Quart. Appl. Math. 14 (1956), 213–214.

L'Auteur démontre qu'il n'existe pas de fonction de courant de la forme $\Psi=r\cdot f(\theta)$, compatible avec les équations de Navier, et qui correspond à un jet issu d'un orifice ponctuel, limité par un cône de révolution.
*R. Gerber* (Toulon).

**Hämmerlin, Günther. Zur Theorie der dreidimensionalen Instabilität laminarer Grenzschichten.** Z. Angew. Math. Phys. 7 (1956), 156–164.

The disturbance equations governing the three-dimensional instability of the laminar boundary layer over a curved surface are usually simplified by a number of approximations in addition to linearization. In a previous paper [J. Rational Mech. Anal. 4 (1955), 279–321;

MR 16, 876], the author showed that on the basis of these simplified equations the parameter $(U_0\vartheta/\nu)(\vartheta/R)^{\frac{1}{2}}$ in the case of neutral stability decreases monotonically to a finite limit when the non-dimensional wave number $\alpha\vartheta\to 0$, where $\vartheta$ and $R$ are the boundary-layer momentum thickness and surface radius of curvature, respectively. In the present paper, he shows that when one additional term involving the curvature is retained in the equations the parameter mentioned above deviates from its previous values for very small values of $\alpha\vartheta$ and becomes infinite when $\alpha\vartheta\to 0$. *D. W. Dunn* (Baltimore, Md.).

**Punnis, B. Zur Differentialgleichung der Plattengrenzschicht von Blasius.** Arch. Math. 7 (1956), 165–171.

For the boundary-layer flow past a flat plate, the solution is required of the equation $f'''+ff''=0$ with the boundary conditions $f(0)=f'(0)=0$, $f'(\infty)=2$. The solution of this boundary-value problem can be obtained from the solution satisfying the initial conditions $f(0)=f'(0)=0$, $f''(0)=1$. The author shows that the latter solution has the singular behavior near a point $\eta^*<0$ given by $f\to 3/(\eta-\eta^*)$ as $\eta\to\eta^*$. He also gives a procedure for calculating $\eta^*$ arbitrarily accurately, and carries this far enough to show that $3\cdot 11\leqq -\eta^*\leqq 3\cdot 13$. *D. W. Dunn.*

**Goldstine, H. H.; and Gillis, J. On the stability of two superposed compressible fluids.** Ann. Mat. Pura Appl. (4) 40 (1955), 261–267.

The authors state that they have extended the known results on the stability of superposed fluids of different densities to include compressibility. Actually, however, the problem they have considered is exactly the same as in Rayleigh's classical investigation [Proc. London Math. Soc. 14 (1883), 170–177=Scientific papers, Cambridge, 1900, pp. 200–207]. This is evident when one observes that their basic equation (eq. 16 of their paper) is the same as that derived by Rayleigh (his eq. 9): the confusion has apparently arisen from not distinguishing between the concepts of a compressible fluid and an incompressible fluid of varying density — a distinction made clear by Rayleigh.] *S. Chandrasekhar* (Williams Bay, Wis.).

**Tchen, Chan-Mou. Stability of oscillations of superposed fluids.** J. Appl. Phys. 27 (1956), 760–767.

L'Auteur étudie la stabilité de la surface de séparation de deux liquides pesants et visqueux, de densités différentes. Les deux couches liquides sont de hauteur infinie et il est tenu compte de la tension superficielle.

Dans les deux cas des longueurs d'onde très grandes et très petites, des approximations successives des solutions de l'équation caractéristique sont étudiées en détail.

Cette équation est ensuite résolue d'une manière approchée par une solution interpolée à partir d'approximations qui sont exactes pour trois zones de valeurs de la longueur d'onde.

Les résultats sont illustrés par des courbes et ils sont comparés à ceux des théories approchées existantes, ainsi qu'aux résultats calculés numériquement à partir de l'équation exacte. *R. Gerber* (Toulon).

**Reid, William H. Two remarks on Heisenberg's theory of isotropic turbulence.** Quart. Appl. Math. 14 (1956), 201–205.

Dans le cas de la turbulence isotrope, l'équation spectrale d'énergie, qui s'écrit, avec les notations classiques

$$(1)\qquad \frac{\partial E(k)}{\partial t}=T(k)-2\nu k^2E(k),$$

relie deux fonctions inconnues $E(k)$, $T(k)$. Pour obtenir une seconde relation entre ces deux inconnues, on introduit en général des hypothèses physiques, dont la plus connue est la théorie du transfert d'Heisenberg, suivant laquelle

$$(2)\quad T(k)=-2\mathrm{K}\frac{d}{dk}\int_k^\infty [E(k')/k'^3]^{\frac{1}{2}}dk'\int_0^k k''^2E(k'')dk''.$$

La théorie d'Heisenberg a reçu diverses critiques. En particulier, la constante K semble dépendre beaucoup du nombre de Reynolds $R_\lambda$. L'auteur veut montrer que, si $S$ est le facteur de distorsion (skewness) de $\partial u_1/\partial x_1$, le rapport $S/\mathrm{K}$ dépend assez peu de $R_\lambda$.

Pour les petites valeurs de $R_\lambda$, une méthode d'approximations successives utilisant comme point de départ les règles de similitude classiques montre que $S/\mathrm{K}$ tend vers 0,78 quand $R_\lambda\to 0$. L'expérience indique que $S$ est de l'ordre de 0,48, ce qui donne pour K une valeur de l'ordre de 0,62.

Pour les grandes valeurs de $R_\lambda$, l'auteur distingue la valeur de $S/\mathrm{K}$ pour $R_\lambda=\infty$, donnée par comparaison avec la théorie de Kolmogoroff, et la limite de $S/\mathrm{K}$ lorsque $R_\lambda\to\infty$. La première valeur serait de l'ordre de 0,60, et la seconde, plus physique, de 1,52.

En conclusion, K semble compris entre 0,20 et 0,62, ce qui constitue un intervalle de variation plus étroit qu'on ne le pensait. *J. Bass* (Paris).

**Fleishman, B. A. Dispersion of mass by molecular and turbulent diffusion: one-dimensional case.** Quart. Appl. Math. 14 (1956), 145–152.

The author considers the one-dimensional equation governing molecular diffusion in the presence of a convection velocity:

$$(1)\qquad s_t-Ds_{xx}=-(sv)_x,$$

where $s(x,t)$=concentration of dispersing matter, $v(x,t)$= convective velocity, $D$=(constant) coefficient of molecular diffusion; the initial distribution $s(x,0)=f(x)$ $(-\infty<x<+\infty)$ is given. Replacing $x, t, s, v$ by dimensionless quantities $\xi, \tau, \sigma, \omega$ (1) is transformed in

$$(2)\qquad \sigma_\tau-\sigma_{\xi\xi}=-\Omega(\sigma\omega)_\xi$$

with the initial condition: $\sigma(\xi,0)=\varphi(\xi)$.

In the first part a method of integration is developped giving the solution of (2) as a power series in $\Omega$:

$$(3)\qquad \sigma(\xi,\tau)=\sum\Omega^i\sigma_i(\xi,\tau);$$

$\sigma_0(\xi,\tau)$ is given in terms of $\varphi(\xi)$ by the Poisson-Fourier integral and $\sigma_i(\xi,\tau)$ in terms of $\sigma_{i-1}(\xi,\tau)$ by Duhamel's formula; a sufficient condition for the uniform convergence of (3) in the strip $\{-\infty<\xi<+\infty, 0\leqq\tau\leqq\tau_0\}$ is given. In the second part it is assumed that the convection velocity $\omega(\xi,\tau)$ is a random function, with mean value zero for all $(\xi,\tau)$. The following results are proved: (a) $\langle\sigma_0\rangle=\sigma_0$; (b) $\langle\sigma_1\rangle=0$; (c) $\langle\sigma_2\rangle$ is expressed in terms of the second order correlation $\langle\omega(\xi_1,\tau_1)\omega(\xi_2,\tau_2)\rangle$; (d) more generally $\langle\sigma_n\rangle$ is expressed in terms of $n$th order correlation of $\omega$. In an example, the dispersion at small time is computed for initial distribution and space correlations of the Gaussian type. *J. Kampé de Fériet* (Lille).

**Rahmatulin, H. A. Foundations of the gas dynamics of interpenetrating motions of compressible media.** Prikl. Mat. Meh. 20 (1956), 184–195.

The author derives general partial differential equations for the interpenetrating motion of compressible ideal

fluids. These equations are, however, mere generalizations of the classical ones. The analysis is based on the continuity equation, which is called the law of Lomonosov. In particular, the author considers rather thoroughly the case of the interpenetrating action of two compressible media. The characteristics pertaining to the appropriate hyperbolical equation are obtained explicitly. Linearizations are also effected. Finally, the author discusses three particular cases, namely, 1) the motion of incompressible fluids, 2) the uniform motion through tubes of variable cross-sections, and 3) the non-uniform flow of incompressible fluids. Numerical results are not given.

*K. Bhagwandin* (Oslo).

★**Patterson, G. N. Molecular flow of gases.** John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London, 1956. x+217 pp. $7.50.

This is an introduction to gas dynamics based on the kinetic theory of gases. The first three chapters give a simple but reasonably complete account of how the Boltzmann equation is derived and solved (to the so-called "first approximation") for pure monatomic gases departing by a small amount from the Maxwell distribution of velocities, leading to the Navier-Stokes equations of motion and to expressions for the variation of viscosity and thermal conductivity with temperature for any given law of intermolecular force. This account will be welcome to those frightened by the more formidable theories described by Chapman and Cowling [The mathematical theory of non-uniform gases, 2nd ed., Cambridge, 1952; for a review of the 1st ed. see MR 1, 187] and by Hirschfelder, Curtiss and Bird [Molecular theory of gases and liquids, Wiley, 1954]. They will find, also, that in the present book the theory is more continuously related to problems of fluid dynamics than in those otherwise more comprehensive works.

In chapter 4 the author gives a too much curtailed account of the evidence regarding intermolecular forces and variation of viscosity and conductivity with temperature. He gives also a quite misleading description of the significance of the so-called "bulk viscosity". A serious student will have to refer to Hirschfelder, Curtiss and Bird for adequate treatments of these topics. The section on shock structure, though good as regards shocks in monatomic gases, suggests quite erroneously (p. 116) that a shock wave in $CO_2$ might be treated theoretically by use of a bulk viscosity as obtained from low-frequency measurements. The author's own photograph of such a shock (fig. 17, p. 150) shows how wrong this suggestion is, but a reader who noticed this discrepancy would be at a loss to explain it on the basis of information given in this book. One needs a clear statement that bulk viscosity is simply a device for representing the effect of relaxation of internal degrees of freedom in such motions as have effective frequencies small compared with the relaxation frequency, as explained at length, for example, in the reviewer's survey article [Surveys in mechanics, Cambridge, 1956, pp. 250–351; MR 17, 1024].

The boundary layer on a flat plate in high-speed flow is then treated, with a good degree of detail, and the chapter ends with a discussion of relaxation effects in which the author follows Bethe and Teller [Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Rep. no. X-117 (1945)] and Kantrowitz [J. Chem. Phys. 14 (1946), 150–164].

The fifth and least chapter is concerned with the flow of rarefied gases, first from the point of view of free-molecule flow. This account is usefully complete, except that it does not include effects of molecular collisions even to a first approximation. Secondly, the author describes the "slip-flow" modifications to boundary layer theory, but here he seriously overestimates the effect of slip on the Blasius value of skin friction on a flat plate by using a Rayleigh-Oseen type of theory of the flow. A consideration of heat transfer with temperature jump is also given.

The book has grown from an advanced lecture course, and there can be no doubt that the general scope of the subject matter is most suitable for a postgraduate course on gas dynamics. Lecturers and students alike will find it useful as giving a nucleus of information in this field, from which they can go on to build by using the extensive bibliographies given at the end of each chapter.

*M. J. Lighthill* (Manchester).

**Ray, M. Velocity and temperature distributions in a forced jet of a compressible fluid.** Bull. Calcutta Math. Soc. 47 (1955), 165–170.

L'auteur étudie l'écoulement plan d'un jet compressible lorsque le fluide est soumis à une force extérieure proportionelle à la masse, au carré de la vitesse et à l'inverse de l'abscisse. Il est ainsi possible, compte tenu des conditions aux limites imposées par le jet, d'intégrér les équations du mouvement. *H. Cabannes* (Québec).

**Kibel', I. A. On adaptation of the motion of the air to the geostrophic motion.** Dokl. Akad. Nauk SSSR (N.S.) 104 (1955), 60–63. (Russian)

L'auteur considère le vent voisin du vent géostrophique. En supposant la terre plate et en linéarisant les équations du mouvement on arrive à un système de 3 équations

$$\frac{du}{dt}=-\frac{\partial H}{\partial x}+lv,\quad \frac{dv}{dt}=-\frac{\partial H}{\partial y}-lu,$$

$$\frac{\partial}{\partial\zeta}\left(\zeta^2\frac{\partial^2 H}{\partial\zeta\partial t}\right)=c^2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right),$$

où $u$ et $v$ les composantes de la vitesse, $H$ le géopotentiel, $\zeta=p/P$ où $p$ la pression et $P$ pression standard au niveau de la mer, $l$ le paramètre de Coriolis, $c^2=\alpha RT$ où $T$ est la température moyenne.

Les conditions aux limites $\partial_2 H/\partial\zeta\partial t=0$ pour $\zeta=1$, et $\zeta\partial^2 H/\partial\zeta\partial t$ bornée pour $\zeta=0$.

Posons

$$u=\frac{\partial\varphi}{\partial x}-\frac{\partial\psi}{\partial y},\quad v=\frac{\partial\varphi}{\partial y}+\frac{\partial\psi}{\partial x}$$

avec $\varphi=\varphi_0$, $\psi=\psi_0$ et $H=H_0$ pour $t=0$ où $\varphi_0$, $\psi_0$ et $H_0$ des fonctions données d'avance de $x$, $y$ et $\zeta$.

En éliminant $\psi$ et $H$ on trouve pour $\varphi$

$$c^2\Delta\varphi+\frac{\partial}{\partial\zeta}\left[\zeta^2\frac{\partial}{\partial\zeta}\left(\frac{\partial^2\varphi}{\partial t^2}+l^2\varphi\right)\right]=0$$

avec $\int_0^1 \varphi d\zeta=0$.

Posons $\Phi=\zeta^{-\frac{1}{2}}\int_0^\zeta \varphi d\zeta$ et $\xi=\ln\zeta^{-1}$. $\Phi$ doit satisfaire à l'équation

$$c^2\Delta\Phi+\left(\frac{\partial^2}{\partial\xi^2}-\frac{1}{4}\right)\left[\frac{\partial^2\Phi}{\partial t^2}+l^2\Phi\right]=0$$

pour $\zeta=0$, $\Phi=0$; pour $\zeta\to\infty$, $\Phi$ est bornée.

Pour trouver la solution on applique la méthode classique de Fourier.

Il est évident que lorsque $\varphi$ et $\partial\varphi/\partial t$ sont différentes de zéro dans un certain domaine limité $\Phi$ tend vers zéro

lorsque le temps $t$ croît et dans ce cas comme il est facile de le voir lorsque $t\to\infty$ le mouvement tend vers un mouvement géostrophique. L'auteur assure qu'on peut trouver les valeurs $H_\infty$ et $\psi_\infty$ directement sans passer par les valeurs intermédiaires. *M. Kiveliovitch* (Paris).

**Byrd, Paul F. Theoretical wave drag of shrouded airfoils and bodies.** NACA Tech. Note no. **3718** (1956), 40 pp.

For airfoils, a shroud consists of flat plates above and below the profile to intercept the waves it produces and reflect them back onto its after parts. For bodies of revolution, the shroud is a circular cylinder. The analytical methods used here are those of familiar linearized supersonic-flow theory and, in particular, Laplace transforms. The wave drag is determined, and body shapes for vanishing wave drag are calculated. In the axisymmetric case this leads to an integral equation, for which certain solutions are found. It is also shown how a body shape can be designed for zero wave drag if a portion of its shape is specified. The drag of these bodies at off-design conditions is discussed. The pressure distribution is also treated.

In an appendix, the inverse Laplace transform of

$$\frac{1}{s}\left[\frac{e^{s}K_1(s)}{\pi e^{-s}I_1(s)}-1\right]$$

is calculated, where $K_1(s)$ and $I_1(s)$ are the usual Bessel functions. *W. R. Sears* (Ithaca, N.Y.).

**Cabannes, H. Tables pour la détermination des ondes de choc détachées.** Rech. Aéro. no. **49** (1956), 11–15.

In cylindrical coordinates $x$, $r$ let $x=x(r)=r^2/2R+\sum_2^\infty \lambda_i r^{2i}/2iR^{2i-1}$ describe the shock wave $S$ ahead of a blunt body of revolution $B$ at zero incidence in a uniform supersonic stream. The author writes the stream function for the non-isentropic flow field behind $S$ as $\psi(x,r)=\sum_{i=1}^\infty\sum_{n=0}^\infty \psi_{2i}^{(n)}(0)r^{2i}x^n/n!$ and also writes as double power series in $x$ and $r^2$ the pressure, density, $u$, and $v/r$, where $u$ and $v$ are $x$ and $r$ components of velocity. For $2i+n\leqq 8$ the coefficients $\psi_{2i}^{(n)}(0)$ have been calculated and depend only on the shock coefficients $R$, $\lambda_2$, and $\lambda_3$. Now suppose $B$ is described by

$$x=h+r^2/2R_{\smile}+\sum_2^\infty \sigma_i r^{2i}/2iR_{\smile}^{2i-1}$$

as well as by $\psi=0$. Then $h/R$, $R_{\smile}/R$, $\sigma_2$, and $\sigma_3$ are readily determined functions of $\lambda_2$ and $\lambda_3$ which have been tabulated for Mach number 2 for use in finding approximately the flow about an arbitrarily prescribed $B$. *J. Giese.*

**Holmquist, Carl O.; and Rannie, W. Duncan. An approximate method of calculating three-dimensional compressible flow in axial turbomachines.** J. Aero. Sci. **23** (1956), 543–556, 582.

The authors assume a steady axisymmetric flow which is compressible but without separation or shock waves. The effect of downstream blades on upstream conditions is neglected. On the assumption of nearly radial blades, the radial force due to the blades is omitted from the equation for the radial component of the motion. On the assumption that the streamlines are known, the continuity equation, the energy equation and the isentropic relations are evaluated immediately upstream and downstream of a stator-rotor stage and between the stator and rotor. A necessary additional equation is obtained by introducing as a known function of the radial distance the rotor blade trailing-edge angle (in the case of the direct problem of flow in a given turbomachine) or the work output of the rotor (in the case of the indirect problem of designing a turbomachine). *M. Marden* (Milwaukee, Wis.).

**Fenain, M.; et Vallée, D. Application de la théorie des mouvements homogènes au calcul des effets de portance pour des ailes en flèche effilées.** Rech. Aéro. no. **50** (1956), 17–25.

The theory of homogeneous supersonic flow fields developed by Germain [Rech. Aéro. no. **7** (1949), 3–16; MR **10**, 492] is applied here to problems of lifting wings having supersonic trailing edges. The results of Germain's work are reviewed briefly. Formulas for lift, drag-due-to-lift, rolling moment and pitching moment are worked out for wings with subsonic leading edges. For wings with supersonic leading edges, the lifting problem is identical with the drag problems worked out in an earlier paper by the same authors [ibid. no. **44** (1955), 9–21; MR **16**, 971]. Application is made to delta wings, and sweptback delta (swallowtail) wings having incidence and parabolic twist (but not camber) within the homogeneous-flow restriction. Results are presented in a series of graphs for delta wings and for one family of sweptback delta wings; relative leading-edge sweep is the abscissa.

Finally, the total drag is evaluated for these two families, the two constants of the parabolic twist distribution being selected for optimum ratio of (drag/lift)$^2$. It is shown that this ratio can always be reduced relative to that of a flat wing by use of this type of twist. The advantage is greater for wings with subsonic than supersonic leading edges. *W. R. Sears* (Ithaca, N.Y.).

**Woods, L. C. Aerofoil design in two-dimensional subsonic compressible flow.** Aero. Res. Council, Rep. and Memo. no. **2845** (1952), 54 pp. (1956).

Previous work by Lighthill [same Rep. no. **2112** (1945)] and the author [ibid. no. 2811 (1950); MR **15**, 360] for incompressible flow is here extended to compressible flow by means of the von Kármán-Tsien approximation. The velocity distribution over only about 80 percent of the chord is specified, leaving the regions near the nose and tail to be determined by the desired nose radius and the closure condition. Some auxiliary matters are also considered, such as the use of this theory to modify pressure distributions, and the approximate calculation of the field of flow surrounding the profile. *W. R. Sears.*

**Martin, A. I. A note on the dividing stream line in hydrodynamics.** Amer. Math. Monthly **63** (1956), 409–410.

By using the elements of vector analysis the author proves the following theorem for three dimensional, unsteady, rotational and compressible flows: In any fluid motion past a fixed solid body, the necessary and sufficient condition that the dividing stream line should meet the boundary surface of the solid at right angles is that any existing vorticity of the fluid at the point of impact should lie along the normal to the boundary surface. The proof applies to obstacles bounded by analytic surfaces only. The case of more than one dividing stream line is also discussed. *E. H. Zarantonello* (Mendoza).

★ **Germain, P. New applications of Tricomi solutions to transonic flow.** Proceedings of the Second U. S. National Congress of Applied Mechanics, Ann Arbor, 1954, pp. 659–666. American Society of Mechanical Engineers, New York, 1955. $9.00.

The author applies to various gas dynamic problems an

ingenious transformation developed by him, Fenain and Liger [C. R. Acad. Sci. Paris **234** (1952), 592–594, 1846–1848; MR **13**, 793; **14**, 920]. If the transformation $3s=2\sigma^{3/2}$, $\omega+is=\tan(\alpha+it)$, $\psi=s^{1/6}\beta^{-1}(t)f(s,\omega)$, $\beta=F(\frac{1}{12},\frac{5}{12},1,\sinh^{-2}2t)$, $\theta=a\alpha$, $\eta=K^{-1}\int_0^t\beta^{-2}(t)\,dt$ is applied to the Tricomi equation $\sigma f_{\omega\omega}+f_{\sigma\sigma}=0$, it becomes the equation for the stream function in the hodograph plane, $k(\eta)\psi_{\theta\theta}+\psi_{\eta\eta}=0$ with $k(\eta)=a^2K^2\beta^4(t)$. The constants $a$ and $K$ can be chosen to correspond to a pressure-density relationship that has many of the properties of actual gases, in particular at low and sonic speeds. The simplest problems carried out are those of flows past straight boundaries. The author determines, explicitly in the hodograph plane, the flows past a wedge with sonic speeds at infinity, the corresponding Helmholtz problem, and the critical jet flow. The flow past a wedge at high subsonic speeds is found approximately. For a more restricted pressure density relation the behavior in the physical plane can be determined exactly and otherwise it is found by using the transonic approximation. A comparison is made with the results of Guderley, Yoshihara and Cole. *C. S. Morawetz* (New York, N.Y.).

**Numerov, S. N. On a method of solution of filtration problems.** Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1954**, no. 4, 133–139. (Russian)

The author discusses filtration in homogeneous furrow-type media, including the effect of evaporation, by means of conformal-mapping theory. The resulting integrals are evaluated exactly as well as approximately. The author's analysis shows that capillary-type media increase the effectiveness of irrigation. The present paper seems to be an extension of the author's earlier results [cf. V. I. Aravin and S. N. Numerov, Theory of motion of liquids and gases in nondeformable porous media, Moscow, 1953]. *K. Bhagwandin* (Oslo).

**Sokolov, Yu. D. On the theory of plane unsteady filtration of ground water.** Ukrain. Mat. Ž. **6** (1954), 218–232. (Russian)

The present work is a continuation of the author's previous investigations [cf. Dopovidi Akad. Nauk Ukrain. RSR **1952**, nos. 4, 5; Ukrain. Mat. Ž. **5** (1953), 159–170; MR **15**, 476]. The work contains a mathematical analysis of the unsteady filtration of ground-water. The non-linear equation (cf. J. Boussinesq, Journ. Math. Pures. Appl., ser. 5, 10, f. 1, 1906) in question is written in the form

$$\frac{\partial h}{\partial t}=\left(\frac{c}{2\mu}\right)\frac{\partial^2 h^2}{\partial x^2}+\frac{w}{\mu},$$

where $h$ denotes the water-level at time $t$, $c$ is the coefficient of filtration, $\mu$ is the coefficient of waterflux, and $w$ is the intensity of infiltration. This equation is subjected to the conditions $h(x,0)=H_0=\text{const}$ $(0\leq x\leq L)$, and $h(0,t)=h_0<H_0$, $(\partial h/\partial x)_{x=L}=0$ ($L$ is the length of the drainage system, measured along the $x$-axis). The problem is decomposed into a depression zone and a layer.

The author solves this problem, approximately, by means of the method of the stationary phase. The analysis is somewhat complicated, but elegant and straightforward. {The approximations seem to be reasonable.} *K. Bhagwandin* (Oslo).

**Sokolov, Yu. D. On an axially symmetric problem of the theory of unsteady motion of ground water.** Ukrain. Mat. Ž. **7** (1955), 101–111. (Russian)

In this paper the author considers the unsteady filtration through a ground layer. The equation and the appropriate initial- and boundary-conditions are virtually the same as in the paper reviewed above. The analysis is also similar. *K. Bhagwandin* (Oslo).

**Ryffert, Halina. Über die Ausbreitung "exponentialbegrenzter" Wellen von endlicher Amplitude.** Acta Phys. Polon. **14** (1955), 435–445. (Russian summary)

Let $\Sigma$ be a surface of revolution about the $x$-axis such that the area bounded by $\Sigma$ on the family of surfaces orthogonal to $\Sigma$ and normal to the $x$-axis increases exponentially with $x$. The equation for the spreading of sound wave through such a horn is derived from the hydrodynamic equations in Lagrangian coordinates. Integration is carried out in successive approximations based upon expansion in powers of $2\pi a/\lambda$, where $a$ is the amplitude of vibration at $x=0$. Attenuation of the fundamental and the harmonics with increasing $x$ is computed and discussed. *R. N. Goss.*

**Heinrich, G.; und Desoyer, K. Hydromechanische Grundlagen für die Behandlung von stationären und instationären Grundwasserströmungen. II.** Ing.-Arch. **24** (1956), 81–84.

The present paper is a logical continuation of the author's previous communication [Ing.-Arch. **23** (1955), 73–84; MR **17**, 205]. In this paper, the authors put forth a general theory for ground-water flow over skeletons (not necessarily homogeneous) with statistical substance distribution. Some particular cases are also noted. However, no existence proofs are given; neither do the authors show how their equations are to be solved. Also, the authors do not seem to be familiar with the extensive literature pertaining to this type of problems [cf. P. Ya. Poluberinova-Kočina, Theory of motion of ground water, Gostehizdat, Moscow, 1952; MR **15**, 71; J. Litwiniszyn, Ann. Soc. Polon. Math. **22** (1949), 185–194; MR **11**, 699]. It shall be of interest to see what numerical results (as promised in the first paper) the derived equations will yield. *K. Bhagwandin* (Oslo).

**Barenblatt, G. I. On some approximate methods in the theory of one-dimensional unsteady filtration of a fluid in the elastic regime.** Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1954**, no. 9, 35–49. (Russian)

The author discusses various aspects of non-uniform homogeneous filtration theory. The analysis proceeds from the equation

$$\frac{\partial p}{\partial t}=a^2x^{-s}\frac{\partial(x^s\partial p/\partial x)}{\partial x},$$

where $p$ denotes the pressure, and $s$ takes the values 0.1, and 2. Numerous integral representations are obtained. The analysis is of an approximate nature. However, the second-order approximation (filtration through a circular layer) is virtually identical with the exact solution. *K. Bhagwandin* (Oslo).

**Barenblatt, G. I. On limiting self-similar motions in the theory of unsteady filtration of a gas in a porous medium and the theory of the boundary layer.** Prikl. Mat. Meh. **18** (1954), 409–414. (Russian)

The present work is a continuation of the author's previous communication [Prikl. Mat. Meh. **16** (1952), 679–698; MR **14**, 699]. Similar methods have been employed before by K. P. Staniukovič [C. R. (Dokl.) Acad. Sci. URSS (N.S.) **48** (1945), 310–312; MR **7**, 446]. The

author obtains limiting self-similar solutions of the equation $\partial p/\partial t=a^2\partial^2 p/\partial x^2$, subject to the conditions $p(x, -\infty)=0$, $p(0, t)=p_0e^{\sigma t}$, where $p$ denotes the pressure. By means of dimensional analysis and some formal transformations, the author obtains the solution in the form $p=p_0e^{\sigma t}f(x[a^2p_0e^{\sigma t}\sigma^{-1}]^{-\frac{1}{2}}$, where

$$(*)\qquad d^2f/d\xi^2+\tfrac{1}{2}\xi df/d\xi-f=0;\ f(0)=1,\ f(\infty)=0.$$

The numerical solution of this equation is given in the paper reviewed below. Other boundary-conditions for the pressure equation are also noted. The author concludes his analysis by obtaining limiting self-similar solutions to the boundary-layer equation [cf. S. Goldstein, Proc. Cambridge Philos. Soc. **35** (1939), 338–340].

*K. Bhagwandin* (Oslo).

**Barenblatt, G. I. On some problems of unsteady filtration.** Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1954**, no. 6, 97–110. (Russian)

In this paper, the author studies the unsteady filtration through a layer. The equation is identical with that given in the preceding paper. The boundary-conditions are $p(x, 0)=0$, $p(0, t)=p_0(t)=\sigma t^\alpha$ ($\alpha, \sigma=$ const $\geqq 0$). The equation for $f(\xi)$ is the same as that of the previous paper, except that the third term in (*) is multiplied by $\lambda=\alpha/(\alpha+1)$. Numerical values are given for different values of the parameter $\lambda$ (for 0.00, 0.05, $\cdots$, 1.00). In particular, the author considers the case of axial-symmetry. Here, too, numerical results are given for the same values of $\lambda$. The flux is also evaluated.

*K. Bhagwandin* (Oslo).

**See also:** Keller, Lewis and Seckler, p. 43; Bergman, p. 73; Bauhuber, p. 74; Stallman, p. 74; Frankl', p. 101; Parker and Krook, p. 93; Tandon, p. 93.

## *Optics, Electromagnetic Theory, Circuits*

**Hopkins, H. H. The frequency response of optical systems.** Proc. Phys. Soc. Sect. B. **69** (1956), 562–576.

The fact that most image forming pencils are oblique to the receiving plane is usually disregarded by most workers in studying diffraction in optical systems. However, it is shown that a correction term is needed to take account of this obliquity, and the size of this term is frequently too large to be neglected in practical optical systems.

The effect of this term in connection with frequency response is considered. It is found that no loss of generality arises from considering unidimensional objects.

*E. W. Marchand* (Rochester, N.Y.).

**Grümm, Hans. Ebene elektrostatische Felder, die eine strenge Berechnung der Elektronenbahnen zulassen.** Ann. Physik (6) **17** (1956), 269–274.

The problem treated is the determination of plane electrostatic fields satisfying the following requirement: Standard electron optical equations which determine the electron trajectories in such fields should be solvable by simple quadratures. The applicability of the method to electron ballistics calculations is not discussed. Criteria are given for the separability of the Hamilton-Jacobi differential equation [cf. Eisenhart, Phys. Rev. (2) **74** (1948), 87–89; MR **9**, 590] and a general formula is derived for the trajectory calculation. Specific examples are given of fields satisfying the basic requirement.

*J. E. Rosenthal* (Passaic, N.J.).

**Grümm, Hans. Rotationssymmetrische elektrostatische Felder, die eine strenge Berechnung der Elektronenbahnen zulassen.** Ann. Physik (6) **17** (1956), 275–280.

The determination of electrostatic fields, which permit the calculation of electron trajectories by simple quadratures — begun in the paper reviewed above — is extended here to fields with rotational symmetry. It is found necessary to impose additional conditions on the field types listed by Staeckel [Math. Ann. **42** (1893), 537–563]. Explicit expressions are given for fields satisfying the basic requirement.

*J. E. Rosenthal* (Passaic, N.J.).

**Lüst, Reimar; Schlüter, Arnulf; und Katterbach, Klaus. Die Bahnen von Teilchen der kosmischen Strahlung im Erdmagnetfeld.** Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. IIa. **1955**, 127–223.

A detailed discussion of the orbits of charged particles in the magnetic field of the earth, with tables containing information on 400 orbits computed on the Göttingen digital computers. Both the general theory of the motion and the methods of approximation and programming are given, and the tables cover momenta ranging from $1.17\times10^{11}$ ev/$c$ to $2.39\times10^9$ ev/$c$.

*H. C. Corben.*

**Chew, G. F.; Goldberger, M. L.; and Low, F. E. The Boltzmann equation and the one-fluid hydromagnetic equations in the absence of particle collisions.** Proc. Roy. Soc. London. Ser. A. **236** (1956), 112–118.

It is often assumed that a fully ionized gas admits a macroscopic description even when the free path for particle collisions is long compared with the distance over which average properties of the gas vary appreciably, provided that the Larmor radius is sufficiently short. The authors test this assumption by trying to derive a set of macroscopic equations from Boltzmann's equation, together with Maxwell's electrodynamic equations. By analogy with the procedure described by Chapman and Cowling [The mathematical theory of non-uniform gases, 2nd ed., Cambridge, 1952; for a review of the 1st ed. see MR **1**, 187] for deriving the equations of hydrodynamics from Boltzmann's equation, the authors expand the Boltzmann function in powers of the Larmor radius, retaining only the first two terms, and then obtain from Boltzmann's equation a sequence of relations between the moments of the velocity distribution. They show that the electronic degrees of freedom can be approximated by a macroscopic current; but the sequence of relations between the moments of the velocity distribution does not terminate, as it does in the hydrodynamic case. The third relation (which contains the equation of energy balance) involves moments of the third order, representing the transport of pressure along magnetic lines of force. (In the hydrodynamic case all the odd central moments vanish, owing to the isotropy of the zero-order Boltzmann function.) If these transport terms are neglected there results a closed set of hydromagnetic equations involving a non-isotropic pressure tensor.

*D. Layzer* (Cambridge, Mass.).

**Parker, Eugene N. Hydromagnetic dynamo models.** Astrophys. J. **122** (1955), 293–314.

In this paper the author describes the following sequence of interactions to serve as a basis for explaining the origin of the earth's magnetic field: A nonuniform rotation generates a toroidal magnetic field from an initial poloidal field; a succession of rising "cyclones" then creates out of the toroidal field, loops of flux in the

meridianal planes; these loops then coalesce and regenerate a poloidal field. Idealized problems incorporating features characteristic of the various links in this sequence of interactions are formulated and solved. In this manner the author seeks to validate his model of a self-exciting dynamo. On the basis of these considerations, the author writes down the following two equations:

$$(*)\quad \frac{\partial B}{\partial t}=H\frac{\partial A}{\partial y}+\frac{1}{4\pi\sigma}\nabla^2 B,\ \frac{\partial A}{\partial t}=\Gamma B+\frac{1}{4\pi\sigma}\nabla^2 A,$$

where $B$ denotes the intensity of the toroidal fields, $A$ the vector potential from which the poloidal field is derived, $H$ a constant characterizing the nonuniform rotation, $\Gamma$ another constant to describe the feedback mechanism and $\sigma$ is the electrical conductivity. Equations (5) allow solutions of the form

$$B=B_0e^{i(\omega t+ky)}\ \text{and}\ A=A_0e^{i(\omega t+ky)},$$

where

$$i\omega=(ikH\Gamma)^{\frac{1}{2}}-k^2/4\pi\sigma.$$

Since $\omega$ has both a real and an imaginary part, these are amplified. A ratio of the amplitudes of the two vector fields is: $A/B=(-i\Gamma/kH)^{\frac{1}{2}}$; this shows that there is a phase shift of 45° between $A$ and $B$. The field thus consists of alternating strands of toroidal field interlaced with alternating loops of poloidal field, the whole traveling along the $y$-axis with velocity $\omega/k$. The author calls these waves migratory dynamo waves. In terms of these waves the author seeks to account for the principal features of the solar magnetic activity. *S. Chandrasekhar.*

**Nag, B. D.; and Sayied, Abdul Maksud. Electrodynamics of moving media and the theory of the Čerenkov effect.** Proc. Roy. Soc. London. Ser. A. **235** (1956), 544–551.

The expression for energy lost in Cerenkov radiation by a charged particle moving in a medium of dielectric constant $\varepsilon$ and permeability $\mu$ is derived by using the Lorentz invariance of Maxwell's phenomenological electrodynamic equations. This does not depend symmetrically on $\mu$ and $\varepsilon$ and the effect of permeability is considered as a subject for experimental verification. {There is a confusing notation in equation (4b) where $\mathbf{v}$, ultimately the velocity of the particle, is used in an expression for a conduction current which is zero in the first part of the argument; cf. equation (6b) and text immediately following it.} *C. Strachan* (Aberdeen).

**Ahiezer, A. I.; Lyubarskiĭ, G. Ya.; and Faĭnberg, Ya. B. On the radiation of charged particles moving through coupled resonators.** Ž. Tehn. Fiz. **25** (1955), 2526–2534. (Russian)

The radiation from a charge moving with a constant velocity in a periodic structure is investigated. The method used is based on regarding the motion of the charge as producing forced oscillations in the structure. The radiation is then associated with resonance between the self-oscillations of the structure and the "external force" connected with the moving charge. The condition for resonance determines the radiation spectrum, and the rate of radiation can be calculated. Several examples are treated. One of these deals with a moving oscillating dipole instead of a charge and involves the Doppler effect. It suggests a method for generating micro-waves. *N. Rosen* (Haifa).

**Tandon, Jagdish Narain. A note on the oscillations of an infinite cylinder subject to radial magnetic field.** Proc. Nat. Inst. Sci. India. Part. A. **21** (1955), 394–403 (1956).

The author discusses the torsional and radial oscillations of a cylindrical mass of highly conducting incompressible fluid in the presence of an infinite axial magnetic line pole. Unfortunately, a trivial error appears to invalidate some of the extensive calculations presented. Equation (24) should read

$$\Delta_1=\frac{\partial^2}{\partial x^2}+\frac{1}{x}\frac{\partial}{\partial x}+\frac{1-\mu^2}{x^2}\frac{\partial^2}{\partial\mu^2}-\frac{\mu}{x^2}\frac{\partial}{\partial\mu},$$

i.e.

$$\Delta_1=\frac{\partial^2}{\partial x^2}+\frac{1}{x}\frac{\partial}{\partial x}+\frac{\partial^2}{\partial\theta^2},$$

and the form of the solutions will then become $\psi=\sum F_n(x)e^{in\theta}$ instead of containing the author's highly suspect Legendre functions. In addition, there remains a question of the validity of the approximation made in (12). At the very least, the form of the modal solution obtained should be checked to demonstrate that the derivatives of the small quantity $h$ may also be neglected. *W. K. Saunders* (Washington, D.C.).

**Karp, S. N.; and Russek, A. Diffraction by a wide slit.** J. Appl. Phys. **27** (1956), 886–894.

The problem of diffraction of scalar waves by an infinite conducting plane with a slit is investigated. Approximate expressions for the near and far fields, taking into account the interaction between the edges, are derived in terms of the well-known solutions for the field produced when an isolated conducting half-plane is excited by (a) a plane wave, and (b) a line source. Results of numerical calculation are given for the case of a plane wave normally incident on the slit. A comparison of transmission coefficients is given. It is found that the new approximate solution agrees well with the exact solution and provides a correction to the noninteraction solution. The accuracy increases with the slit width, so that the result is useful in the range where interaction cannot well be neglected but where the exact solution converges so slowly that computation is impracticable. (From the author's summary.) *A. E. Heins* (Pittsburgh, Pa.).

**Gajewski, Ryszard. On transient radiation of a dipole inside a wave guide. I.** Acta Phys. Polon. **15** (1956), 25–41. (Russian summary)

It is desired to determine the electromagnetic field of a dipole inside a waveguide of arbitrary shape and constant cross-section under the assumption that the dipole moment has the form $\mathbf{M}f(t)$, where $\mathbf{M}$ is independent of time. Conditions which must be satisfied by $f(t)$ in order to insure uniqueness of the solution are derived, and the function $f(t)=1(t)(1-e^{-\alpha t})\sin\omega_0 t$, where $1(t)$ is the Heaviside unit function, is shown to satisfy those conditions. With this choice of a particular function, formulas for the field are developed, first by finding the field due to vibration of a single frequency and then that arising from superposition of waves of the entire frequency spectrum. *R. N. Goss* (San Diego, Calif.).

**Parker, E. N.; and Krook, M. Diffusion and severing of magnetic lines of force.** Astrophys. J. **124** (1956), 214–231.

It is well-known that in a perfectly conducting fluid the

magnetic lines and tubes may be regarded as frozen into the fluid. This implies that a tube can never break in two, and two distinct tubes can never coalesce. If the electrical conductivity is finite, however, the magnetic lines diffuse through the fluid, and it is possible for initially distinct loops to coalesce and for a loop to detach itself from a twisted tube. The authors study these processes in detail for the case of a fluid at rest and for tubes whose radii are small compared with their radii of curvature. Under these conditions the magnetic field satisfies a diffusion equation, whose solutions can be expressed in terms of the well-known solution for diffusion from a line source.

*D. Layzer* (Cambridge, Mass.).

**Marziani, Marziano. Su alcuni problemi dell'elettrodinamica stazionaria dei superconduttori ciclici.** Ann. Univ. Ferrara. Sez. VII. (N.S.) 4 (1954–1955), 81–90.

$S_0$ is an $(n+1)$-ply connected superconductor and $S_\infty$ the rest of space. In a stationary state, the magnetic vector $\mathbf{H}$ and the current density $\mathbf{j}$ satisfy (with $c$, $\lambda$ given constants)

$c$ rot $\mathbf{H}=\mathbf{j}$, $c$ rot $\lambda\mathbf{j}=-\mathbf{H}$ in $S_0$; div $\mathbf{H}=0$, rot $\mathbf{H}=0$ in $S_\infty$.

The total currents are

$$i_r=\int_{\sigma_r}\mathbf{j}\times\mathbf{n}d\sigma_r\ (r=1,2,\cdots,n),$$

taken over barriers $\sigma_r$ which make $S_0$ simply connected, and the fluxoids are

$$\phi_r=c\lambda\int_{l_r}\mathbf{j}\times dP_0+\int_{\Sigma_r}\mathbf{H}\times\mathbf{n}\,d\Sigma_r\ (r=1,2,\cdots,n),$$

where $l_r$ is a curve enclosing a hole in $S_0$ and $\Sigma_r$ a surface bounded by $l_r$ ($\times$=scalar product). There are two problems: find $\mathbf{j}$ and $\mathbf{H}$ given (i) the total currents, (ii) the fluxoids. In each case $\mathbf{H}$ is to vanish at infinity and certain continuity conditions are to be satisfied on the surface $\sigma$ which separates $S_0$ and $S_\infty$. The author shows that the first problem is reducible to the second. In dealing with the second problem, use is made of a vector potential $\mathbf{A}$ satisfying rot $\mathbf{A}=\mathbf{H}$, div $\mathbf{A}=0$, and a multiple-valued superpotential $\chi$ satisfying grad $\chi=\mathbf{A}+c\lambda\mathbf{j}$. The problem is then to find $\mathbf{A}$ and $\chi$ to satisfy

$$\Delta\mathbf{A}-\beta^2\mathbf{A}=-\beta^2\ \mathrm{grad}\ \chi \text{ in } S_0\ (\beta^{-2}=c^2\lambda);\ \Delta\mathbf{A}=0 \text{ in } S_\infty,$$

with $\partial\chi/\partial n=\mathbf{A}\times\mathbf{n}$ on $\sigma$, continuity of the first normal derivative of $\mathbf{A}$ across $\sigma$, and with jumps in $\chi$ across $\sigma_r$ of amount $\phi_r$. Finally the problem becomes that of solving the integral equation

$$\mathbf{A}(P)=-\frac{1}{4\pi}\int_{S_0}\frac{\beta^2}{|P-P_0|}[\mathbf{A}(P_0)-\mathrm{grad}\ \chi(P_0)]dS_0(P_0),\ PCS_0,$$

and the existence of a solution is established under certain restrictive conditions. Uniqueness of solution is a consequence of an earlier paper by the author [Boll. Un. Mat. Ital. (3) 9, 409–412 (1954); MR 16, 775]. *J. L. Synge.*

**See also**: Epstein, p. 30; Van Regemorter, p. 102; Chandrasekhar, p. 86; Pao, p. 87; Ferreira, p. 96.

## *Thermodynamics and Heat*

**Jaeger, J. C. Conduction of heat in an infinite region bounded internally by a circular cylinder of a perfect conductor.** Austral. J. Phys. 9 (1956), 167–179.

A solid that occupies the region $r>a$, where $r$ is the usual cylindrical coordinate, has initial temperature zero. The cylindrical space $r<a$ is filled with a substance that is a perfect conductor of heat. There is a thermal contact resistance at the surface $r=a$ such that the flux across that interface is proportional to the difference between the temperatures of the core $r<a$ and the temperature of the solid at that interface. Cases in which the core has a uniform initial temperature, and in which heat is generated within the core, are considered here. The solutions of the boundary value problems written in terms of integrals involving Bessel functions, are used together with dimensionless variables to tabulate and to show graphically the temperature of the core at all times. In one case the temperatures within the solid $r>a$ are also displayed graphically. *R. V. Churchill.*

**Reid, Walter, P. Heat flow in a half space.** Quart. Appl. Math. 14 (1956), 206–208.

Temperature $u(x,t)$ is given for all $x>0$ at $t=0$. Radiation occurs for $t>0$ from surface $x=0$ to a thin slab which re-radiates to surroundings at known temperature. Solution for $u$ for $t>0$ is found by a combined use of Laplace and Fourier transforms. *D. R. Bland.*

**Lessen, M. Note on the symmetrical property of the thermal conductivity tensor.** Quart. Appl. Math. 14 (1956), 208–209.

Author shows that the assumption of symmetry is unnecessary in certain applications because the anti-symmetrical part of the tensor can make no contribution.

*D. R. Bland* (London).

**Buff, Frank P. Curved fluid interfaces. I. The generalized Gibbs-Kelvin equation.** J. Chem. Phys. 25 (1956), 146–153.

The transition zone at the interface separating two fluid states is investigated using two different methods, first a molecular or perhaps more correctly a hydrodynamic approach and secondly by means of thermodynamic arguments. An external field producing a force proportional to the density (gravitational or centrifugal forces) is included. Both methods yield the same generalized Gibbs-Kelvin equation that gives the pressure difference between the two fluids in terms of the surface tension, curvature and external forces. Various assumptions and resulting limitations of the theory are discussed.

*G. Newell* (Providence, R.I.).

## *Quantum Mechanics*

**Zaitsev, G. A. Tensors which are characterized by two real spinors.** Soviet Physics. JETP 2 (1956), 240–246.

Translation into English by M. Hamermesh of the Russian article by G. A. Zaĭcev in Ž. Eksper. Teoret. Fis. 29, 166–175 (1955) reviewed in MR 17, 330.

**Zaitsev, G. A. Real spinors in curvilinear coordinates and pseudo-Riemannian space.** Soviet Physics. JETP 2 (1956), 290–296.

Translation into English by D. E. Spencer of the Russian article by G. A. Zaĭcev in Ž. Eksper. Teoret. Fis. 29, 345–353 (September, 1955) reviewed in MR 17, 564.

**Narumi, Hajime. On the eigenvalue problem in terms of a complete set of the Casimir operators.** J. Phys. Soc. Japan 11 (1956), 786–792.

While the connection between Newtonian mechanics and quantum mechanics is undoubtedly substantial its exact mathematical nature remains but poorly defined. On the plane of analysis the best-worked field is that of the W.K.B. approximation and its generalizations, but even here only the one-dimensional case has been examined in detail. On the level of functional analysis and group theory there exists a theory of contact transformation groups but the part which is common to both kinds of mechanics depends as yet on a small body of particular examples. The author is concerned with the latter theory. His major point concerns the use of a set of operators of the type employed by H. Casimir [Akad. Wetensch. Amsterdam, Proc. 34 (1931), 844–846] for the study of semi-simple groups. This set of operators plays the role of a complete set of commuting observables in the author's considerations. Applications are given to known examples to show how the degree of representations is determined. {It would be desirable from the point of view of extension of the general theory if the author's work were brought into connection with the representation theory of continuous groups as developed by I. M. Gel'fand and his collaborators [Gel'fand and Naĭmark, Trudy Mat. Inst. Steklov. 36 (1950); MR 13, 722].}

*E. L. Hill* (Minneapolis, Minn.).

**Landé, Alfred. Déduction de la théorie quantique à partir de principes non-quantiques.** J. Phys. Radium (8) 17 (1956), 1–4.

This represents a continuation of the author's book "Foundations of quantum theory···" [Yale Univ. Press, 1955; MR 17, 437]. The relations among transition probabilities obtained previously are regarded as determining a general metric, and from this the author arrives at the form of the probability amplitudes as complex numbers. From the postulate of constant probability density in phase space the probability amplitude connecting two conjugate variables is obtained, which is essentially equivalent to obtaining their commutation relation.

*N. Rosen* (Haifa).

**Tharrats, Jesus. Sur les ondes singulières de la mécanique quantique.** Acta Salmant. Ser. Ci. (N.S.) 1 (1956), no. 7, 19 pp.

Il s'agit d'un essai de théorie des particules élémentaires. La particule est considérée d'abord „comme un modèle d'univers d'Einstein" puis „comme la singularité d'un champ caractérisé par un nombre infini dénombrable de propriétés", ce qui amène à associer à chaque particule élémentaire une fonction d'onde interne complexe $u(z)$ de la variable complexe $z$, holomorphe sauf en un point $z_0$ „représentant la localisation ponctuelle de la particule". „C'est donc dans le domaine complexe qu'il faut chercher les propriétés de la particule élémentaire". L'auteur se propose alors de généraliser la théorie causale de de Broglie [La Physique quantique restera-t-elle déterministe? Gauthiers-Villars, Paris, 1953; MR 16, 1183] et Bohm [Phys. Rev. (2) 85 (1952), 166–179, 180–193; MR 13, 709, 710] en l'étendant au domaine complexe. Reprenant un aspect de la théorie de la double solution de de Broglie [loc. cit.], l'auteur associe à une solution „régulière" $\psi(x)=\sum_m^\infty A_k(x-x_0)^k$ de l'équation de Schroedinger la fonction d'onde interne complexe singulière $u(z)=\sum_{-\infty}^{+\infty} A_m(z-z_0)^m$ coïncidant avec $\psi(z)$ au temps $t=0$. Ceci conduit à donner à la particule une vitesse complexe, dont la partie réelle est précisément la vitesse donnée par la formule de guidage de de Broglie. L'étude de l'onde interne singulière $u(z)$ permet à l'auteur de prévoir une vie moyenne, une valeur du spin et une valeur de la masse pour les différents états de la particule élémentaire. Pour des raisons parfois typographiques, le lecteur pourrait avoir quelque peine à suivre le développement de la pensée de l'auteur.

*D. Rivier* (Lausanne).

**Basu, D.; and Sural, D. P. Compton scattering of light by electron.** Indian J. Phys. 30 (1956), 265–272.

The Compton scattering of light by an electron was calculated long ago and the cross section of the process is given by the famous Klein-Nishina formula. It is not the aim of the present paper therefore to give a more accurate expression for the cross section. In fact, the semirelativistic approximation used in this paper yields a less accurate answer than the Klein-Nishina formula. On the other hand, however, it has the advantage that the relativistic correction to the classical Thomson formula and the correction term due to spin can be separated, while the Klein-Nishina formula, being based on the covariant Dirac theory, gives a combined expression for the relativistic and spin corrections. The result of the present paper is that the relativistic correction decreases the cross section, while the spin effects increase it but by a smaller amount. This conclusion confirms the result obtained by Booth and Wilson [Proc. Roy. Soc. London. Ser. A. 175 (1940), 483–518], who got a smaller cross section for scattering of light by a scalar meson that the Klein-Nishina formula gives for electrons. The procedure used in the present paper is a straight forward second order perturbation theory, using a Hamiltonian which contains a Pauli type spin term.

*M. J. Moravcsik.*

**Destouches, Jean-Louis. Fonctions indicatrices de spectres.** J. Phys. Radium (8) 17 (1956), 475–479.

To a real or complex spectrum of an operator in Hilbert space the author associates a complex function — the indicatrix — which vanishes at precisely the points of the spectrum. In case the spectrum is totally discontinuous without accumulation point in the finite plane, the resulting indicatrix is an entire function. All the problems of quantum mechanics which have been solved explicitly give rise to entire functions of genus zero or one.

*A. J. Coleman* (Toronto, Ont.).

**Tsuneto, Toshihiko; Hirosige, Tetu; and Fujiwara, Izuru. Relativistic wave equations with maximum spin two.** Progr. Theoret. Phys. 14 (1955), 267–282.

The relativistic wave equations with maximum spin 2 in the canonical form are investigated under the assumption that each matrix involved is a direct sum of two commuting Duffin-Kemmer operators. Sets of tensor equations are obtained, and these are reduced to simple sets of equations corresponding to definite values of mass and spin. It is found that there are two possible mass values and three possible spin values: 2, 1 and 0. In the case of spin 1, in addition to the usual equations of the vector meson, equations of a different type are obtained. Neither the total energy nor the total charge of the whole field is positive definite.

*N. Rosen* (Haifa).

**Ivanenko, D.; and Mirianašvili, M. Non-linear generalization of the Dirac spinor equation.** Dokl. Akad. Nauk SSSR (N.S.) 106 (1956), 413–414. (Russian)

The problem is considered of two quantized Dirac

fields with a suitable interaction between them. On the assumption that one of these fields is in the vacuum state, it is shown that to a certain approximation the other field satisfies a nonlinear generalization of the Dirac equation.
*N. Rosen* (Haifa).

**Roberts, K. V. The electromagnetic field in a Lagrangian quantum theory.** Phil. Mag. (7) **46** (1955), 941–950.
The author postulates certain rules for deriving the propagation functions for a system of interacting fields from the Lagrangian density. He then suggests that it might be simpler to give up the usual canonical formalism of quantum field theory, and obtain the matrix elements for physical processes directly from his postulates. As an example, the electromagnetic field is discussed in some detail. *S. N. Gupta* (Detroit, Mich.).

**Borgardt, A. A. On the theory of meson fields. III. Conservation of physical quantities.** Ž. Eksper. Teoret. Fiz. **30** (1956), 330–333; supplement to 30, no. 2, 6. (Russian. English summary)
A formal analysis of the representation of meson fields by means of sets of anticommuting matrices. The conservation laws are written down in this representation. This work is a continuation of earlier papers by the author [Dokl. Akad. Nauk SSSR (N.S.) **78** (1951), 1113–1114; Ž. Eksper. Teoret. Fiz. **24** (1953), 24–32, 284–292] which are inaccessible to this reviewer. *F. J. Dyson.*

**Borgardt, A. A. Matrix aspects of the theory of bosons.** Ž. Eksper. Teoret. Fiz. **30** (1956), 334–341; supplement to 30, no. 2, 6. (Russian. English summary)
Connections are established between the author's representation of meson fields by means of anti-commuting matrices [see the preceding review] and the usual Duffin-Kemmer matrix formalism. *F. J. Dyson.*

**Moldauer, P. A.; and Case, K. M. Properties of half-integral spin Dirac-Fierz-Pauli particles.** Phys. Rev. (2) **102** (1956), 279–285.
The Dirac-Fierz-Pauli (DFP) equations for half-integral spin particles are treated in this paper by eliminating their extraneous components. First the field-free DFP equations are written down in general tensorial form and also in a form analogous to the Dirac equation [the Rarita-Schwinger form, Phys. Rev. (2) **60** (1941), 61]. Following this Lagrangian densities are given from which these equations can be derived. These Lagrangian densities form a one-parameter infinite set. The same parameter also appears when the interaction with electromagnetic field is introduced, which therefore can also be carried out in an infinite number of ways. Next the reduction of the equations is performed by a decomposition of the wave function into parts which transform according to various representations of the three dimensional rotation group. The Hamiltonian form of the equations for the independent components are obtained. The detailed reduction is carried out for spins of 3/2 and 5/2. Finally the DFP equations in an electromagnetic field are studied in the non-relativistic limit using an expansion in the reciprocal mass of the particle. The first order of this expansion gives the magnetic dipole moments, and the second order the electric quadrupole moment. It is shown that for any spin the gyromagnetic ratio is independent of the above mentioned parameter in the Lagrangian density. For a spin of 3/2 the same is true for the quadrupole moment, but for a spin of 5/2 the latter depends slightly on the parameter.
*M. J. Moravcsik* (Upton, N.Y.).

**Ferreira, Erasmo M. Radiation field of an oscillating dipole. I.** An. Acad. Brasil. Ci. **28** (1956), 83–94.
The quantum radiation field produced by an alternating current in a closed circuit is computed under the assumption that the frequency of the field oscillators is not near the resonant frequency of the source, consequently that the reaction of the field on the source is negligible. The resulting average field is compared with the classical field. Both steady-state and non-stationary descriptions of the field are given. *R. N. Goss* (San Diego, Calif.).

**Nagel, B. A remark on quantum electrodynamics with non-vanishing photon mass and Lamb shift calculations.** Nuovo Cimento (10) **3** (1956), 496–498.
It is possible to treat quantum electrodynamics as a limiting case of a neutral meson theory with a vanishing mass. It is then usually assumed that the contribution to various processes of the longitudinal mesons vanishes in the limit. The purpose of the note is to show that this statement is misleading, and that the longitudinal mesons do give a non-vanishing contribution to the cross-section for "brems-strahlung" with emission of soft photons. *D. Rivier* (Lausanne).

**Helmis, G. Zur Theorie der Störstellenelektronen. I. Optische Übergänge.** Ann. Physik (6) **17** (1956), 356–370.
A theory of the optical transitions of electrons bound to imperfections is developed. First the Hamiltonian is written down for the coupled electron-phonon system in terms of creation and annihilation operators. The lattice vibrations are described in terms of harmonic oscillators with an arbitrary $\omega$ spectrum. The interaction between electrons and the phonons is linear in the phonon operators. The calculation is carried out using a Dirac perturbation method, but first the Hamiltonian is transformed so that the perturbation is small. A special case of this transformation corresponds to the Born-Oppenheimer approximation. A thorough discussion of this transformation is promised to be given in a forthcoming second part of this paper. After the transformation the interaction between the crystal and the electromagnetic field depends only on the phonon operators. Then the Condon-approximation is investigated and it is found to be quite good in the neighbourhood of an absorption maximum and to deviate more and more from the more precise values as one leaves these maxima. The relationship of the present calculation to previous work by Huang and Rhys [Proc. Roy. Soc. London. Ser. A. **204** (1950), 406], by Lax [J. Chem. Phys. **20** (1952), 1752–1760], and by O'Rourke [Phys. Rev. (2) **91** (1953), 265–270] is also discussed. Finally the various moments of the absorption curves are calculated. *M. J. Moravcsik.*

**Meyer, Klaus. Zur Durchführung des Tomonaga-Verfahrens in einer skalaren Modelltheorie.** Ann. Physik (6) **18** (1956), 104–112.
Heber [Ann. Physik (6) **15** (1955), 157–173; MR **17**, 566] has applied Tomonaga's method of intermediate coupling to a scalar meson field coupled to a non-relativistically moving source. This problem is reinvestigated and the ground state energy $E_0$ as well as the effective mass $\kappa^*$ of the "dressed" source particle are computed. As upper cut-off on the meson momenta the author chooses the

bare mass of the source, $\varkappa$. In the weak coupling limit the results agree with second order perturbation theory, but otherwise nothing can be said about the accuracy of the result, except that the computed $E_0$ and $\varkappa^*$ are upper and lower limits, respectively, which is implied in the variational method employed. *F. Rohrlich.*

**Landau, L. D.; Abrikosov, A.; and Halatnikov, L. On the quantum theory of fields.** Nuovo Cimento (10) 3 (1956), supplemento, 80–104.

This is a condensed translation of a series of papers by the same authors [Dokl. Akad. Nauk SSSR (N.S.) **95** (1954), 497–500, 773–776, 1177–1180; **96** (1954), 261–264; MR **16**, 315, 316; and A. A. Abrikosov, A. D. Galinin and I. M. Halatnikov, ibid. **97** (1954), 793–796; MR **16**, 317].
*F. J. Dyson* (Princeton, N.J.).

**Infeld, L.; and Plebański, J. On an operational method of solving the Klein-Gordon equation.** Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 215–219.

The advanced and retarded Green functions for the Klein-Gordon equation are obtained in two forms involving transcendental functions of operators $n$ and $m$ acting on a delta function. The operator $n$, for example, is the square root of the mass constant minus the Laplacian. Of the infinity of ways of specifying the square root, one is arbitrarily defined in terms of a Fourier Analysis of the wave function. {The advantages of the present approach, over that of § 20 of the standard Russian text-book by Ivanenko and Sokolov [Classical theory of fields, Gostehizdat, Moscow-Leningrad, 1949; MR **13**, 95; **16**, 203] which obtains a more explicit result, are not apparent to the reviewer.} *A. J. Coleman* (Toronto, Ont.).

**Ozaki, Shoji. On quantum electrodynamics without subsidiary conditions.** Progr. Theoret. Phys. **14** (1955), 511–522.

Dans la représentation de Heisenberg, la forme proposée de l'électrodynamique quantique adopte à la place de l'équation d'onde usuelle $\Box A_\mu(x)=-j_\mu(x)$ l'équation $(\delta_{\mu\nu}\Box-\partial_\mu\partial_\nu)A_\nu(x)=-j_\mu(x)$ en écartant la condition de Lorentz $\partial_\nu A_\nu(x)=0$ qui permet de passer d'une équation à l'autre. Suit la décomposition de $A_\mu(x)$ en une partie transversale $\mathfrak{A}_\mu(x)=T_{\mu\nu}A_\nu(x)$ au moyen de l'opérateur de projection

$$T_{\mu\nu}=\delta_{\mu\nu}+(n_\mu\Box-\partial_\mu\partial)\{(\Box-\partial^2)^{-1}+\alpha\delta(\Box+\partial^2)\}n_\nu \\ -(n_\mu\partial+\partial_\mu)\{(\Box-\partial^2)^{-1}+\alpha\delta(\Box+\partial^2)\}\partial_\nu$$

où $n_\mu{}^2=-1$ (vecteur temporel), $\partial\equiv n_\mu\partial_\mu$, $\alpha$ nombre quelconque, et une partie longitudinale contenant le potentiel de Coulomb. L'hamiltonien dans la représentation d'interaction s'obtient à partir des relations de commutation dans cette représentation. Par une transformation unitaire, on ramène les équations de mouvement à la forme usuelle, ce qui démontre l'équivalence des formulations proposée et usuelle. Aucun commentaire n'est fourni concernant la définition du vide. *D. Rivier.*

**Gotô, Ken-iti. On a regular formulation of quantum field theory. I. Non relativistic theory.** Progr. Theoret. Phys. **15** (1956), 167–177.

This is an attempt to reformulate the quantum theory of fields by using the notion of distribution as introduced by L. Schwartz [Théorie des distributions, t. I, II. Hermann, Paris, 1950, 1951; MR **12**, 31, 833]. The paper is concerned only with non-relativistic field theory. The starting point is to rewrite the Lagrangian function by replacing the usual field products by direct products acting themselves on suitable distributions. The equations of motion, the canonical formalism and the quantization are then obtained in the usual way, but contain direct products and distributions instead of simple products. Whereas then all equations are well defined and unambiguous, in the limit where the distributions are replaced by $\delta$-functions of Dirac's type one gets the usual formulation with all divergence difficulties. Application is made to the "non-relativistic electrodynamics" in Fermi's version, with subsidiary condition acting on the state vector. It is shown how for any process the proposed formulation leads to unambiguous finite results in place of the so called renormalized results (obtained from well manipulated infinities.) *D. Rivier* (Lausanne).

**Chisholm, J. S. R. The $S$-matrix for neutral PS-PV meson-nucleon interaction.** Phil. Mag. (8) **1** (1956), 338–344.

Nucleon and pseudoscalar neutral meson fields with pseudovector interaction are considered by means of the Dyson [Phys. Rev. (2) **73** (1948), 929–930] transformation. This gives an exponential of the meson field in the interaction Hamiltonian [cf. Glauber, ibid. **84** (1951), 395–400]. The classification of the graphs representing this interaction is achieved by defining the 'skeleton' of a graph, formed by rubbing out internal meson lines until only one line joins each pair of vertices that were joined originally. 'Similar graphs' have the same skeleton. Rules for the application of these ideas to the evaluation of $S$-matrix terms are given and illustrated by examples.
*C. Strachan* (Aberdeen).

**Brodskiĭ, A.; Ivanenko, D.; and Korst, N. Difference of masses of elementary particles.** Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 1192–1195. (Russian)

The question of the mass difference between two particles of the same type, one of which is charged and the other neutral, is considered. Expressions for this mass difference are obtained for the case of particles of zero spin (as, for example, charged and neutral $\pi$-mesons) and for the case of particles of spin $\frac{1}{2}$ (as, for example, the proton and the neutron). Comparison is made with experiment. *N. Rosen* (Haifa).

**Lieb, E. H. Second-order radiative corrections to the magnetic moment of a bound electron.** Phil. Mag. (7) **46** (1955), 311–316.

The second order radiative corrections are calculated to the magnetic moment $\mu$ of an electron bound by an atomic electrostatic field (potential energy $V$). The method used is that employed previously for the hyperfine structure by Kroll and Pollock [Phys. Rev. (2) **86** (1952), 876–888]. The result is

$$\frac{\Delta\mu}{\mu}=\frac{26}{15\pi}\alpha\frac{\langle V\rangle}{mc^2}.$$

When applied to the outer electron in the alkali atoms this effect is too small to account for the observed dependence upon atomic number of the electron moment. When applied to hydrogen $\Delta\mu/\mu\simeq-(2/\pi)\alpha^3$ and may be of importance with further improvement of present experimental accuracy. *F. Rohrlich.*

**Sokolov, A. A.; and Kerimov, B. K. On the theory of scattering of Dirac particles, taking damping into account.** Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 961–964. (Russian)

Calculations are carried out for the scattering of a Dirac particle by a central force, based on the use of wave functions involving damping [W. Heitler, Proc. Cambridge. Philos. Soc. **37** (1941), 291–300; MR **4**, 95; A. H. Wilson, ibid. 301–316; MR **4**, 95; A. A. Sokolov, Acad. Sci. USSR. J. Phys. **5** (1941), 231–237; MR **4**, 96]. The equations obtained are applied to the case in which the scattering potential is proportional to the Dirac delta-function. *N. Rosen* (Haifa).

**Matthews, P. T.; and Salam, Abdus. On the spin of the $\theta^0$-meson.** Phil. Mag. (7) **46** (1955), 150–154.

The authors investigate the possibility of deciding whether the $\theta^0$ meson is a scalar or a vector particle. Using each possibility, they calculate the cross sections for the reactions $\theta^0\rightarrow\pi^++\pi^-+\gamma$ and $\pi^-+p\rightarrow\Lambda^0+\theta^0$, and compare their results with the experimental data. *S. N. Gupta* (Detroit, Mich.).

**Migdal, A. B.; and Polievktov-Nikoladze, N. M. Quantum kinetic equation for double collisions.** Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 233–235. (Russian)

The motion of particles under the influence of a large number of scattering centers is investigated. By the use of the density matrix an approximate kinetic equation is obtained, in which multiple collisions are neglected. *N. Rosen* (Haifa).

**Banerjee, C. C. Polarisation in p-p scattering.** Indian J. Phys. **30** (1956), 292–298.

The author investigates the polarization in proton-proton scattering with the hypothesis that the particles obey a Dirac equation with a spherically symmetric exponential potential. The characteristic spin-orbit type interaction needed to produce polarization thus arises in a natural fashion in the non-relativistic reduction the of Dirac equation. The calculation is performed under the assumption that only the $P$ wave produces polarization. This phase shift is calculated in Born approximation. The results agree broadly with the experimental energy and angular distribution. At 230 Mev the magnitude of the "left-right asymmetry" obtained is about two-thirds the experimental value. The author hopes that the introduction of tensor forces will raise the theoretical result. No discussion is given as to how well the potential which is used here fits the other p-p scattering data. *R. Arnowitt* (Syracuse, N.Y.).

**Zyryanov, P. S.; and Eleonskiĭ, V. M. Linearization of the Hartree equations.** Ž. Eksper. Teoret. Fiz. **30** (1956), 592. (Russian)

A short note discussing the application of the Hartree equation to the study of the collective motions of a set of particles. The case of a system with nearly uniform density is studied by linearization of the equation in terms of parameters describing the deviation from constant density. When these equations are solved in terms of plane waves a dispersion relation is found which is comparable with that found by Bohm and Pines [Phys. Rev. (2) **92** (1953), 609–625, 626–636]. It is indicated that the method can be applied to the study of surface oscillations of nuclei. The extension to a special case of the Hartree-Fock equations is mentioned briefly. *E. L. Hill* (Minneapolis, Minn.).

**Laing, E. W. Spin orbit coupling and the mesonic Lamb shift.** Phil. Mag. (7) **46** (1955), 106–108.

This paper contains an improvement over the earlier work of Chisholm and Touschek [Phys. Rev. (2) **90** (1953), 763–765] on the field theoretical derivation of the spin-orbit coupling, required for the nuclear shell model. Using the charge symmetrical pseudoscalar meson theory with the pseudoscalar coupling, the author shows that the self-energy corrections for a nucleon moving in a scalar potential well lead to a spin-orbit coupling, which is of the right sign and the right order of magnitude. *S. N. Gupta* (Detroit, Mich.).

**Kalitzin, Nikola St. Über die fünfdimensionale Kerntheorie und eine neue Lösung der Dipolschwierigkeit.** C. R. Acad. Bulgare Sci. **7** (1954), no. 3, 1–4 (1955). (Russian. German summary)

By setting up Maxwell equations in a five-dimensional space and assuming a suitable dependence of the field variables on the fifth coordinate, the author obtains equations for a vector meson field and from them an expression for the interaction between two nucleons. The imposition of an additional condition eliminates the dipole term, with its troublesome singularity, from the interaction. *N. Rosen* (Haifa).

**Martin, André. Une démonstration élémentaire des inégalités causales de Wigner.** C. R. Acad. Sci. Paris **243** (1956), 22–23.

Wigner [Phys. Rev. (2) **98** (1955), 145–147; MR **17**, 114] has shown that the derivative of the scattering phase shift with respect to energy must exceed a certain limit if the interaction between scattered particle and scattering centre vanishes beyond a certain distance. His inequalities are here derived from the Schrödinger equation. *C. Strachan* (Aberdeen).

**Ham, Norman S.; and Ruedenberg, Klaus. Electronic interaction in the free-electron network model for conjugated systems. I. Theory.** J. Chem. Phys. **25** (1956), 1–13.

During the last few years a modified form of perturbation method for the study of the electronic states of certain classes of complex organic molecules, which is known as the "metallic" or "free-electron network" model, has been developed. In this model orbital functions are chosen which permit an electron to circulate around closed paths in the molecule. The paths lie geometrically along the valence bond directions, but the paths for particular electronic states are chosen to correspond to empirically known properties of the molecule. The authors study the electronic configuration interaction in this model. This involves the evaluation of certain integrals, the properties of which are determined to some extent by the symmetry properties of the molecules. Procedures for the evaluation of some of the integrals by identifying them with empirical data on the spectra of the molecules are discussed. *E. L. Hill* (Minneapolis, Minn.).

**Putnam, C. R. Note on the many-particle problem.** Quart. Appl. Math. **14** (1956), 101–102.

Consider the simple Schrödinger Hamiltonian

$$H=\Big(\sum_{k=1}^{N} H_k\Big)+\sum_{1\leq j<k\leq N}\frac{2}{r_{jk}},\ H_k=-\nabla_k^2-\frac{2Z}{r_k}$$

for $N$ orbital electrons in an atom with stationary nucleus and atomic number $Z$, spins and the antisymmetry

of the electron state function required by the exclusion principle being neglected. By a simple calculation, generalizing that of Kato for $N=2$ [Trans. Amer. Math. Soc. **70** (1951), 212–218; MR **12**, 781], the author shows here that if $Z \geqq (5/8)N(N-1)$, then the least real number in the spectrum of $H$ is actually in the point spectrum, being an eigenvalue.

*F. H. Brownell* (Seattle, Wash.).

**Hosemann, R.; und Bagchi, S. N. Quantenmechanik als Beugungsphänomen von Führungswellen. I. Die Eigenschaften eines spinfreien Teilchens, definiert durch vier Prinzipe.** Z. Physik **142** (1955), 334–346.

Les auteurs proposent une nouvelle „Théorie des ondes pilotes" destinée à fondre dans un ensemble harmonieux la mécanique classique et la mécanique quantique des particles sans spin. Mécaniques classiques et quantiques apparaissent comme deux cas limites différents de la nouvelle théorie, qui s'appuye sur quatre principes „classiques". I. Le principe de relativité restreinte d'Einstein. II. Le principe de Newton-Minkowski généralisé (définissant le concept de corpuscule à partir de l'amplitude d'une onde pilote $\varepsilon$ scalaire dans l'espace — temps. III. Le principe de de Broglie généralisé (reliant la quantité de mouvement totale de la particule à la phase de l'onde pilote). IV. Le principe de Huyghens généralisé (exprimé sous la forme d'une équation de continuité pour une „densité de quantité de mouvement du champ" (Wellenfeldimpulsdichte). De ces quatre principes découle une équation d'onde généralisée pour l'onde pilote $\varepsilon$, équation non linéaire. En l'absence de champ extérieur, cette équation se réduit à l'équation linéaire de Gordon-Klein, et dans le cas où l'amplitude de l'onde ne dépend pas du temps, l'équation se réduit à l'équation de Schrödinger relativiste indépendante du temps. Enfin lorsque l'amplitude de l'onde pilote satisfait l'équation des ondes électromagnétiques, l'équation différentielle pour la phase de l'onde pilote coïncide avec celle de Hamilton-Jacobi de la mécanique classique. Les auteurs utilisent un formalisme qui permet de considérer des solutions discontinues aux équations de mouvement. A ce point près, la „Théorie des ondes pilotes" n'est essentiellement qu'une nouvelle version des théories „à paramètres cachés" édifiées notamment par de Broglie [La physique quantique, restera-t-elle déterministe? Gauthier-Villars, Paris, 1953; MR **16**, 1183] et D. Bohm [Phys. Rev. (2) **85** (1952), 166–179, 180–193; MR **13**, 709, 710].

*D. Rivier.*

**Hosemann, R.; und Bagchi, S. N. Quantenmechanik als Beugungsphänomen von Führungswellen. II. Die Prinzipien der klassischen Mechanik und Quantenmechanik als Entartungsfälle.** Z. Physik **142** (1955), 347–362.

Dans cette seconde partie, les auteurs développent les conséquences de leur théorie de l'onde pilote généralisée. C'est d'abord une mécanique relativiste du point qui fait intervenir une „force de diffraction" (Beugungskraft), qui n'est autre que la force quantique de de Broglie [C. R. Acad. Sci. Paris **233** (1951), 641–644]. Puis une mécanique de Lagrange généralisée se distinguant essentiellement de la forme traditionnelle par une contribution proprement ondulatoire dans la lagrangienne. Il en est de même pour la forme hamiltonienne de la mécanique généralisée, laquelle conduit à considérer l'énergie de la particule comme répartie entre celle-là et le champ d'onde qui la guide, cette répartition variant avec le temps dans le cas général. Selon la théorie de l'onde pilote généralisée, il faut, plutôt qu'une seule particule et sa trajectoire, considérer un ensemble de particules définies par un ensemble d'ondes pilotes $\varepsilon_k=\varepsilon \exp(i\delta_k)$ ne différant que par un facteur de phase aléatoire $\delta_k$. Les conditions initiales fixent une densité d'occupation $\gamma^2\varepsilon\varepsilon^*$ à un instant donné d'une région donnée de l'espace (KAB normierung). Si $\gamma^2$ est une distribution de Dirac, on retombe sur le problème de mécanique du point modifié par l'existence de l'onde pilote (Individualnormierung). Certains autres choix particuliers de $\gamma^2$ réduisent le problème à celui de la mécanique ondulatoire d'une particule. Dans un dernier alinéa, les auteurs montrent comment la théorie de l'onde pilote généralisée entraîne l'existence de relations d'incertitudes du type de celles de Heisenberg, cela d'une manière parfaitement analogue à celle utilisée par la mécanique ondulatoire traditionnelle. Ces relations d'incertitudes généralisées se réduisent à celles de Heisenberg lorsqu'il s'agit de particules qui ont ou bien une masse nulle ou bien qui sont observées de distances supérieures au rayon nucléaire ($r\sim 10^{-13}$ cm).

*D. Rivier.*

**Hosemann, R.; und Bagchi, S. N. Quantenmechanik als Beugungsphänomen von Führungswellen. III. Maxwellsche Strahlungstheorie und Schrödingersche Wellenmechanik als Entartungsfälle. Kernphysik und experimentelle Prüfung der neuen Theorie.** Z. Physik **142** (1955), 363–379.

Dans ce troisième mémoire, les auteurs étudient les conséquences de leur „théorie des ondes pilotes" pour certains cas particuliers. D'abord dans le cas d'une particule de masse de repos nulle et de charge nulle (photon de spin zéro): la théorie prévoit une masse „effective" de repos non nulle et une „force de diffraction" également différente de zéro dans les régions où il y a diffraction de Fresnel, ces mêmes grandeurs s'annulant dans les régions à diffraction de Fraunhofer. La théorie prévoit aussi que si dans une région de Fraunhofer les photons se meuvent avec la vitesse $c$, il n'en est plus de même dans la région de Fresnel où la vitesse des photons est inférieure à $c$. La théorie prévoit qu'un ensemble de photons de spin zéro (KAB Normierung) se comporte énergétiquement comme le champ de radiation électromagnétique, avec cependant cette anomalie caractéristique que dans une région de Fresnel, l'intensité lumineuse n'est plus proportionnelle au carré du module de l'amplitude de l'onde (pilote). Dans le cas des électrons périphériques de l'atome, certaines conditions se trouvent réalisées en 1-ère approximation (orthogonalité des quadrivecteurs potentiel électromagnétique et gradient de l'amplitude de l'onde pilote, vitesse des électrons suffisamment inférieure à celle de la lumière, et facteur de diffraction suffisamment petit), conditions qui font dégénérer la nouvelle théorie dans la mécanique ondulatoire non relativiste habituelle. Par contre, dans le cas plus général de champs de force extérieurs quelconques, la densité de courant de probabilité de présence des particules ne satisfait pas à l'équation de continuité, car il y a alors échange continuel d'énergie entre les particules et leur onde pilote. Dans le cas plus général encore où les vitesses sont de l'ordre de celle de la lumière et où le facteur de diffraction est notoirement plus grand que 1, (c'est le cas d'un électron situé à une distance du noyau inférieure à $10^{-13}$ cm), l'équation de Schrödinger (relativiste) n'est plus valable d'après la nouvelle théorie, qui la remplace par l'équation non linéaire de l'onde pilote. Les auteurs terminent en proposant une série d'expériences cruciales destinées à

prouver leur théorie, en particulier la mesure de la vitesse de la lumière dans une région de diffraction de Fresnel.

*D. Rivier* (Lausanne).

**Hosemann, R.; und Bagchi, S. N. Korrespondenz des Kepler-Problems der klassischen Mechanik und der Schrödingerschen Eigenfunktionen des Wasserstoffatoms.** Z. Physik **145** (1956), 65–82.

Il s'agit d'une discussion du problème de Képler — plus exactement de celui de l'atome d'hydrogène non relativiste — à la lumière de la „théorie générale des ondes pilotes" proposée par les auteurs dans les oeuvres analysées ci-dessus. En étendant aux fonctions d'ondes $\psi$ (de la mécanique ondulatoire) des états liés une décomposition proposée par Sommerfeld [Atombau und Spektrallinien, Bd. II, 2. Aufl., Vieweg, Braunschweig, 1951, S. 118–123] dans le cas des états ionisés, les auteurs démontrent que, d'après leur théorie, 1°) les électrons liés décrivent des trajectoires qui ne peuvent être des ellipses que dans l'approximation des nombres quantiques élévés (NQE). 2°) le carré $\psi\psi^*$ du module de la fonction d'onde décrivant un état stationnaire ne peut pas être une mesure de la densité ou de la probabilité de présence de l'électron. 3°) $\psi$ elle-même n'est pas une onde pilote, mais la superposition de 2 ondes pilotes singulières $\varepsilon_1$, $\varepsilon_2$ guidant l'électron dans son mouvement, l'une de l'aphélie au périhélie et l'autre du périhélie à l'aphélie. La démonstration est donnée pour l'approximation NQE. {Il faut remarquer, concernant les points 1) et 2), que les auteurs admettent a priori l'existence de trajectoires pour l'électron et que dans la mesure où l'expérience a vérifié que $\psi\psi^*$ donne bien en général la probabilité de présence de l'électron, le résultat 2) semble infirmer la théorie proposée.}

*D. Rivier* (Lausanne).

**Leech, J. W. The influence of retardation on the London-van der Waals forces.** Phil. Mag. (7) **46** (1955), 1328–1336.

Casimir and Polder [Phys. Rev. (2) **73** (1948), 360–372] calculated the retardation correction to the electrostatic interaction between two identical neutral atoms. Computing first the distortion of the radiation field due to the presence of one atom and then the interaction between the distorted field and the second atom, they used a perturbation calculation which is unsymmetrical between the atoms. They found that retardation modifies the $R^{-6}$ dependence of the interaction energy to $R^{-7}$ at large $R$. The present author solves the same problem in standard Schrödinger perturbation theory in which the two atoms enter symmetrically and finds an asymptotic $R^{-3}$ law instead of $R^{-7}$. Arguments are presented as to why the two methods should not necessarily agree and why the present method should yield the correct result.

*F. Rohrlich* (Iowa City, Iowa).

**See also:** Finkelstein, p. 100.

## *Relativity*

**Finkelstein, David. Internal structure of spinning particles.** Phys. Rev. (2) **100** (1955), 924–931.

Le but du mémoire est l'étude des structures internes possibles des particules élémentaires sous les conditions d'invariance relativistes et de covariance quantique. Grâce à une méthode fondée sur la théorie des groupes continus, il est possible de classer les types de particules (systèmes rigides) suivant le nombre [$n$] de degrés internes de liberté. En mécanique classique (non relativiste) il existe 3 classes de systèmes rigides représentées par le point matériel [0] le dipôle [2] et le rotateur [3]. Seul le rotateur possède des états de spin $\frac{1}{2}\times\hbar$. En mécanique relativiste, il faut avant toutes choses définir la notion de rigidité: un système est défini comme rigide si: a) l'espace de configuration est homogène vis à vis des rotations et des translations spatiotemporelles; b) aucun des points de cet espace ne reste fixe lors d'une translation spatiotemporelle. Les classes de structures rigides essentiellement distinctes sont alors au nombre de 11; soit 9 classes simples caractérisées par les nombres de degrés internes de liberté: [0], [2], [3], [3'], [4], [4'], [4''], [5] et [6] et 2 classes [$3_f$] et [$5_f$] ($0\leqq f<\pi$) composées d'une infinité continue à 1 paramètre de structures différentes. Seules les structures [4], [5] et [6] possèdent des états de spin $\frac{1}{2}\times\hbar$. Du fait que l'énergie totale du système dépend de son état de structure interne, la particule décrite par le système possède automatiquement un spectre de masse. Celui-ci est calculé dans le cas, simple mais de maigre intérêt physique, d'une particule du type de Gordon-Klein.

*D. Rivier.*

**Zmorovič, V. A. On the restricted relativistic problem of two bodies.** Ukrain. Mat. Ž. **6** (1954), 105–113. (Russian)

As the restricted relativistic problem of two bodies the author considers the problem of motion of a particle in a field of force in the Euclidean space which is central symmetric and in which the velocity of motion can not be greater than a positive constant $c$. A problem of this kind was firstly formulated by Levi-Civita [Der absolute Differentialkalkül, Springer, Berlin, 1928, S. 182–186], where he had proposed as Lagrangian for such a field the following function

$$L^*=c^2\left(1-\sqrt{\left(1-\frac{v^2}{c^2}-\frac{2\Phi(r)}{c^2}\right)}\right).$$

With $c\to+\infty$ we obtain the classical Lagrangian function $L=\frac{1}{2}v^2+\Phi(r)$. The author states that this problem of Levi-Civita was never fully analyzed and therefore he analyzes it at first in full extent. He finds that in this motion too the orbits in the form of conic sections are possible but with other law of the motion on the path as in the classical case. He proposes then three new particular forms for the Lagrangian function which all are of the form $L^*=\frac{1}{2}v^2+\Phi+o(1/c^2)$. These new cases are not analyzed. There are given only hints that this analysis can be carried out without difficulty by means of the method applied to classical case by B. H. Fradlin [Izv. Kiev. Politehn. Inst. **9** (1949); **10** (1950)]. Finally, the author proves the equivalence of this problem to the classical for a particularly chosen potential function.

*T. P. Anđelić* (Belgrade).

**Durand, Émile. Définition d'un élément de volume invariant pour un système en mouvement.** C. R. Acad. Sci. Paris **243** (1956), 354–357.

Reasons are given for adopting the formula

$$dx_1dx^2dx_3/\sqrt{(1-\beta^2)}$$

for the volume element of a moving body.

*A. G. Walker* (Liverpool).

**Petrov, A. Z. On spaces of maximal mobility which define a gravitational field.** Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 905–908. (Russian)

La variété Riemannienne $V_4$ avec l'élément linéaire $ds^2=g_{\alpha\beta}dx^\alpha dx^\beta$ définie le champ gravitationel si dans chaque point l'élément $ds$ détermine la géométrie de Minkowski et le tenseur fondamental $g_{\alpha\beta}$ et le tenseur de Ricci satisfont à l'équation $R_{\alpha\beta}=\varkappa g_{\alpha\beta}$ où $\varkappa$ est une constante arbitraire. L'auteur a montré dans un travail récent qu'il existe seulement trois types de ces espaces. Dans cet article on étudie la question si tous les types d'espaces possèdent le groupe des mouvements d'ordre maximum. On construit d'abord un espace à 6 dimensions, localement centre-affin, formé par les bivecteurs et dans lequel il existe les tenseurs symétriques $R_{ab}$, $g_{ab}$ ($a, b=1, \cdots, 6$) avec dét $|g_{ab}|\neq 0$. Trois types du champ de la gravitation correspondent aux trois caractéristiques de la matrice $\|R_{ab}-\lambda g_{ab}\|$. On peut obtenir deux parties univoques de chaque caractéristique et pour cette raison on peut dire que nous avons trois couples de racines complexes conjugées de l'équation (1) $|R_{ab}-\lambda g_{ab}|=0$. En partant de l'équation de mouvement $v_{(\alpha\beta)}=0$ et des conditions d'integrabilité de cette équation on démontre les théorèmes: 1) Si les racines de l'équation (1) sont diverses, les espaces du premier et du second type avec le groupe de mouvements d'ordre maximum sont symétriques. Les espaces du troisième type ne peuvent jamais être symétriques. 2) Si les racines de l'équation (1) sont égales, les espaces du premier type avec le groupe de mouvements d'ordre maximum possèdent la courbure constante. Si deux racines de l'équation (1) sont différentes, les espaces possèdent le groupe de mouvements à 6 paramètres. 3) L'espace du second type avec le groupe de mouvements d'ordre maximum possèdent le groupe transitif à 6 paramètres. 4) Les espaces du troisième type possèdent le groupe de mouvements d'ordre $m\leqq 4$. L'auteur donne aussi les formules pour $ds^2$ des espaces mentionés.

*F. Vyčichlo* (Prague).

**Frankl', F. I. On gravitational waves and the motion of a gas in strong varying gravitational fields.** Kirgiz. Gos. Univ. Trudy Fiz.-Mat. Fak. **1953**, no. 2, 47–65. (Russian)

The equations of general relativity theory for the gravitational field in empty space are investigated, and it is shown that they possess solutions representing gravitational waves, propagating with the speed of light. The one-dimensional case is treated by the method of characteristics. The dimension is extended to the more general field equations describing the presence of a gas in addition to the gravitational field. In this case the characteristics correspond to the propagation of gravitational waves, the propagation of sound waves and the motion of the gas particles. *N. Rosen* (Haifa).

**Helliwell, J. B. Disturbances in an expanding universe.** Ann. Astrophys. **19** (1956), 19–33. (Russian summary)

Small perturbations of the metric and energy-tensor of an expanding universe are considered and the equation of propagation of harmonic disturbances is established. In static universes of positive curvature the surface of a spherical disturbance can differ from the tangent plane at a point halfway to the antipodal point of the centre, but no such position exists in a space of negative curvature. The extension of these results to non-static universes is indicated. The stability of local disturbances is also discussed, showing under what conditions stable disturbances can occur in static models and also in non-static ones. *G. C. McVittie* (Urbana, Ill.).

**Širokov, M. F. A general theory of relativity or theory of gravitation?** Ž. Eksper. Teoret. Fiz. **30** (1956), 180–184. (Russian)

Arguments are presented against the point of view, advocated by V. A. Fock in various publications, that the general theory of relativity is only a theory of gravitation. The author maintains that it has a more general significance: by means of covariant laws it describes the dependence of the properties of space and time on matter and its motion. *N. Rosen* (Haifa).

**Graiff, Franca. Sul tensore elettromagnetico in una recente teoria unitaria.** Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **19(88)** (1955), 833–840.

Let $\Gamma_\iota$ be the torsion vector (assumed to be zero by Einstein in his unified field theory), $R_{\iota k}$ the contracted curvature tensor and

$$R_{\iota k}{}^* \overset{\text{def}}{=} R_{\iota k}+\Gamma_{\underset{+}{\iota};k}.$$

Einstein's field equations are

$$R_{(\iota k)}=0,\quad \partial_{[j}R_{\iota k]}=0,\quad \Gamma_\iota=0,$$

while Finzi [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **14** (1953), 581–588; MR **15**, 563] assumes $\Gamma_\iota\neq 0$ and

$$(1)\ R^*_{(\iota k)}=0,\quad (2)\ \partial_{[j}R^*_{\iota k]}=0,\quad (3)\ R^{*\iota k}{}_{|k}=0.$$

Here the indices are raised by means of the tensor $g^{\iota k}$ inverse to $g_{\iota k}$ and the bar denotes the covariant derivative with respect to the symmetric part of the connection. The author proves that if (3) holds then

$$F_{\iota k}=\tfrac{1}{2}e_{\iota krs}R^*_{[pq]}g^{rp}g^{sq}-\partial_{[k}\Gamma_{\iota]}$$

is an irrotational tensor. He computes also its first and second approximation. *V. Hlavatý*.

**See also:** Günther, p. 79; Nag and Sayied, p. 93; Tsuneto, Hirosige and Fujiwara, p. 95.

## *Astronomy*

**Wilkens, Alexander. Untersuchungen zur Beschleunigung des Enckeschen Kometen.** Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1955**, 285–302 (1956).

The observational evidence of the secular acceleration of Emcke's comet is based on a comparison of observations with the computed path of the comet, obtained by numerical integration of the perturbations in the elements. Questioning the reliability of these perturbation calculations, the author uses a numerical-analytical method to evaluate the long-period perturbations in the mean motion of the comet associated with the approximate commensurability 18/5 between the periods of Jupiter and the comet. The result is negative [as was to be expected]; this confirms the reality of the principal observed deviation from gravitational theory in the comet's motion.

*D. Brouwer* (New Haven, Conn.).

**Ogorodnikov, K. F. On the dynamics of the local system.** Leningrad. Gos. Univ. Uč. Zap. **136.** Ser. Mat. Nauk **22** (1950), 3–9. (Russian)

By an application of the kinematics of centroids and by taking into account the observed singularities of motion of stars in the local system the author concludes that this system describes a circular orbit about the center of Galaxy; and, in addition, rotates about it with an angular velocity which is 3.3 times greater than that of the Galaxy.

*E. Leimanis* (Vancouver, B.C.).

**van Regemorter, Henri. Calcul des raies d'absorption dans les spectres stellaires.** C. R. Acad. Sci. Paris **242** (1956), 2507–2509.

The equivalent width of an absorption line formed in a model stellar atmosphere is given by an integral that depends in a complicated way upon several functions of frequency and depth. One can always evaluate such an integral by numerical quadrature, but as one usually wants to have equivalent widths for a number of lines methods have been devised for simplifying the calculation. These involve the use of auxiliary functions that can be tabulated once and for all for a given model atmosphere. Unsöld's [Physik der Sternatmosphären, 2nd ed., Springer, Berlin, 1955] method of "weighting functions", originally devised for weak lines has been extended by Pecker [Ann. Astrophysique **14** (1951), 115–151 to any absorption line for which the Doppler width and the damping width do not vary appreciably over the region in which the line is formed. The author extends Pecker's method by allowing for the variation of the damping width with depth. To use the revised method one needs to know the numerical values of an additional auxiliary function, which, however, is the same for all models. The author promises to publish a table of this function in a future paper.

*D. Layzer* (Cambridge, Mass.).

★**Mertens, Robert. Bijdrage tot de theorie van de veelvoudige verstrooiing van deeltjes. [Contribution to the theory of the multiple scattering of particles.]** Simon Stevin, vol. 30, supplement. De Natuur- en Geneeskundige Vennootschap, Gent, 1954. 111+vii pp.

In solving the integro-differential equations which occur in the theory of radiative transfer (or neutron diffusion) in plane parallel slabs, one generally replaces the integrals by Gaussian sums and solves the resulting set of ordinary linear equations. The solutions obtained by these methods are not very accurate near the boundary since the derivatives of the source functions have logarithmic singularities here. To improve the accuracy it has been suggested (apparently first by J. Yvon) that the range of integration from $-1$ to $+1$ by split into two (between $+1$ to 0 and 0 to $-1$) and that the relevant functions in these two intervals be expanded in terms of complete sets of polynomials orthogonal in the respective intervals. These polynomials are the usual Legendre functions $P_n$ but for the arguments $2\mu-1$ and $2\mu+1$ respectively. This method has recently been discussed [cf. Gross and Ziering, Astrophys. J. **123** (1956), 343–352; MR **17**, 1256] but the present paper (which is earlier than other more recent publications on the same subject) is the most complete discussion of this method. The solutions for the cases of isotropic scattering and scattering in accordance with Rayleigh's phase function are obtained and compared with the exact results when these are available. The appendix to this paper contains an exhaustive discussion of the polynomials $P_n(2\mu\pm1)$, the equations and recurrence relations satisfied by them, and their orthogonality and other integral properties. The integrals

$$\mathfrak{P}_{k,i}^{\pm}=\int_0^1 P_k(\mu)P_i(2\mu\pm1)d\mu$$

which are needed in the theory are discussed at some length: the recurrence relations satisfied by them are derived and tables for $k$ ($\leq 5$) and $i$ ($\leq 6$) are provided.

*S. Chandrasekhar* (Williams Bay, Wis.).

**Jefferies, J. T. Radiative transfer with central sources of non-uniform directional intensity.** Proc. Phys. Soc. Sect. B. **69** (1956), 577–582.

The mathematical problem discussed here is the solution of

$$\nabla^2 J-3\varkappa^2\lambda J=-12\pi\varkappa^2 P,$$

where $\varkappa$, $\lambda$ are constants and $P$ is a given source function. The first cases discussed correspond to point or line sources with arbitrary directional dependence. The author then goes on to discuss composite models having different properties inside and outside a sphere (cylinder) concentric with an isotropic point(line) source. The analysis is quite straightforward.

*E. T. Copson.*

**See also:** Moser, p. 40; Pál, p. 76; Lawden, p. 81; Chandrasekhar, p. 86.

## OTHER APPLICATIONS

### *Games, Economics*

**Richardson, Moses. On finite projective games.** Proc. Amer. Math. Soc. **7** (1956), 458–465.

J. von Neumann and O. Morgenstern [Theory of games and economic behavior, 2nd ed., Princeton, 1947, p. 469; MR **9**, 50] have given a seven-person game in which the minimal winning coalitions have the same intersection structure as do the lines of the seven-point plane projective geometry, and they point out that this is the only finite projective geometry for which the lines have the intersection structure of the minimal winning coalitions of a simple game. Shapley [Lectures on $n$-person games, Princeton University Notes] has extended the definition of simple games by permitting blocking coalitions, losing coalitions whose complements are also losing. The author restricts his attention to simple games in this latter sense corresponding to finite projective planes, $PG(2, p^n)$. He proves a number of interesting theorems concerning the possible sizes of the blocking coalitions of which some are the following.

Let $B$ be a blocking coalition and let $|B|$ be the number of players in $B$. If $p^n>2$, there exist blocking coalitions $B$ for which $|B|=2p^n$. If $p^n>2$, then for all $B$ we have $|B|>1+p^n$. (That every two-dimensional finite projective game has a solution is a corollary to this theorem.) In $PG(2, 3)$ the minimum number of elements in a blocking coalition is 6. If $p^n>3$, there exists a $B$ with $|B|=2p^n-1$. If $d$ is a divisor of $n$, $1\leq d<n$, then there exists a $B$ with $|B|=2p^n-p^d+1$.

*E. D. Nering* (Tucson, Ariz.).

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